Dynamic Analysis of Uncertain Structures Using Imprecise Probability

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Existence of Uncertainty

Uncertainty in any physical structure comes from several factors, including:

- -Modeling Errors
- -Discretization Errors
- -Computational Errors (truncation)
- -Material Abnormalities/Defects
- -Load-Structure Interaction Complexities

Uncertainties are unable to be handled with deterministic analyses and thus require some form of uncertain analyses to properly analyze

Uncertainty Modeling

There are two distinct types of uncertainty:

-Aleatoric: Due to inherent randomness in a system. Always present and irreducible.

-Epistemic: Due to lack of knowledge, modeling errors, and/or insufficient data to accurately reflect the system. Is reducible with further data acquisition and model updating.

Uncertainty Analysis Paradigms

There are likewise two distinct types of uncertainty analysis:

-Isomorphic: Analyses which consider only either probabilistic or possibilistic methods of analysis. Allows for uncertainty in the parameter being modeled.

-Polymorphic: Analyses which generate probabilistic information on possibilistic bounds. Allows for uncertainty in both the parameter and the model itself.

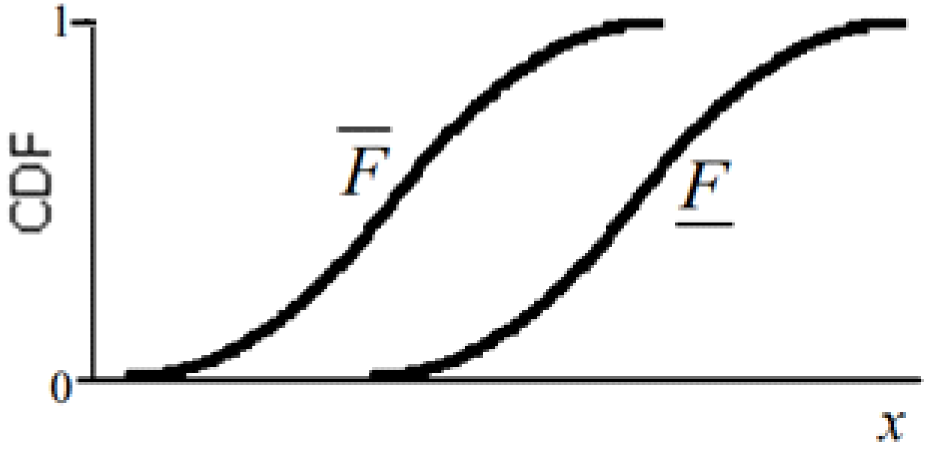
Presentation Outline

- Imprecise Probability
- P-box Structures
- Uncertainty Analyses Application to Dynamic Analysis
  - Conventional Deterministic Response Spectrum Analysis (RSA) Review
  - Developed Imprecise Probability Response Spectrum Analysis (IPRSA)
- Numerical Examples
- Conclusions

Imprecise Probability

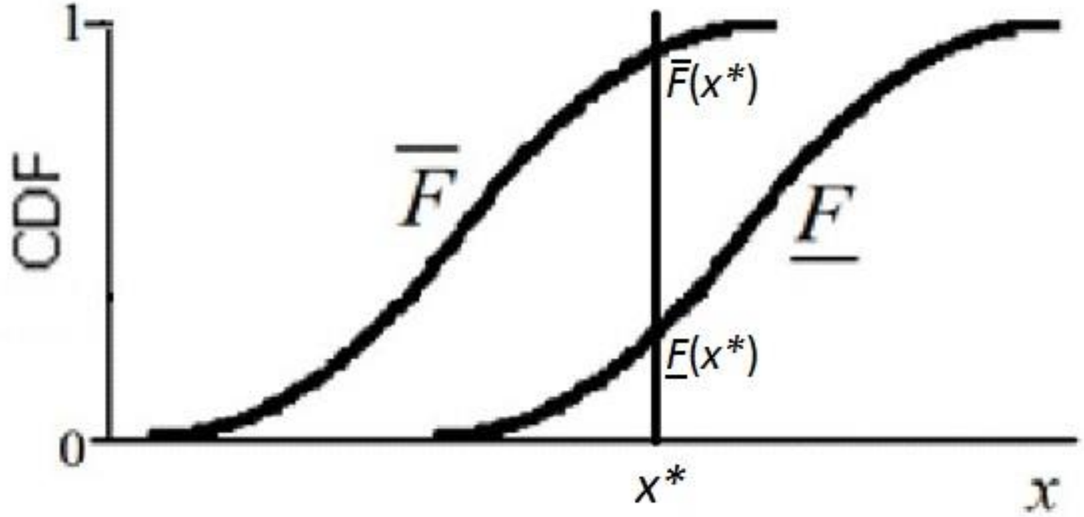
- Framework for handling incomplete information with uncertain PDF or CDF

- Involves setting bounds on CDF based on deterministic or non-deterministic parameters (mean, variance, etc.)



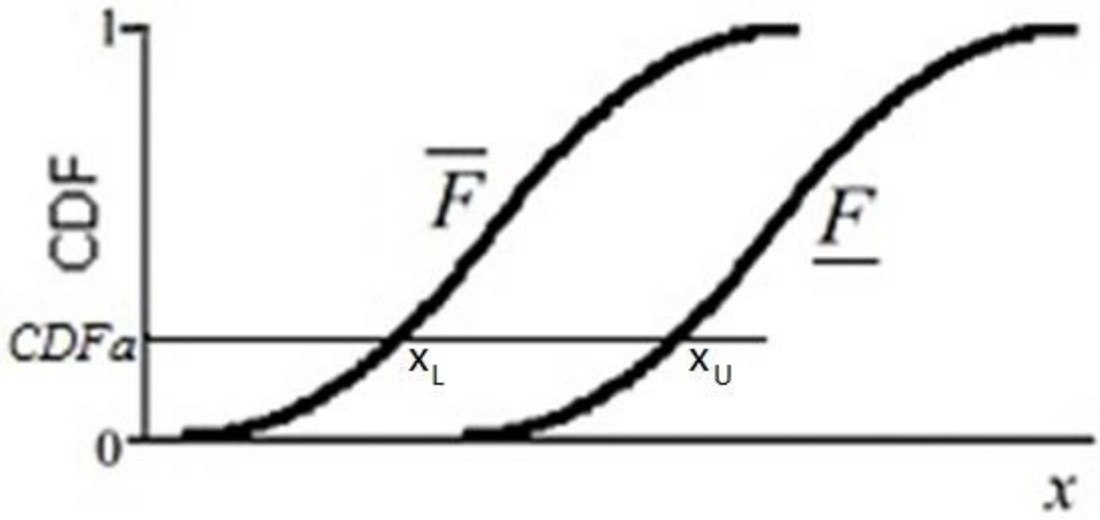
Imprecise Probability

- Drawing a vertical line:
  - $\overline{F}_i$ represents the *UPPER* bound and $\underline{F}_i$ represents the *LOWER* bound on CDF for known x*



Imprecise Probability

- Drawing a horizontal line:
  - $\overline{F}_i$ represents the *LOWER* bound and $\underline{F}_i$ represents the *UPPER* bound on RV x for known CDF

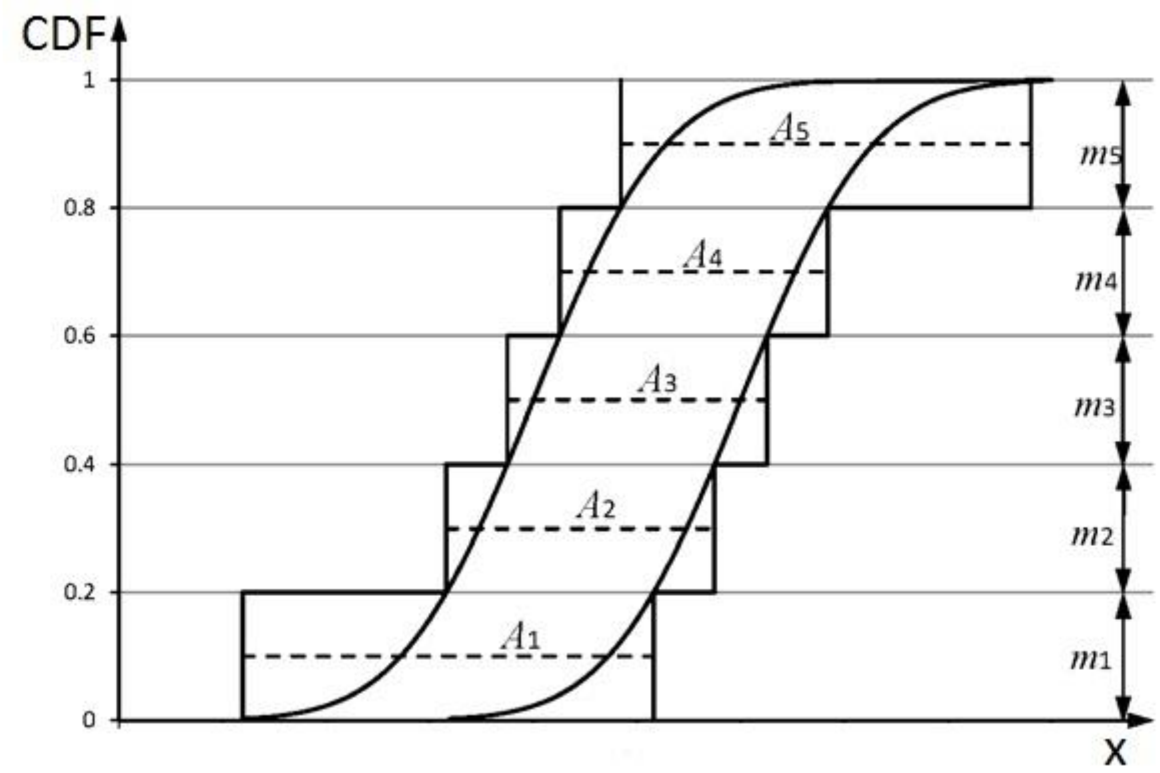


Imprecise Probability

- Using imprecise probability requires choice of whether to model CDF or RV x with uncertainty
  - Chose to model CDF values as exact such that all error is in the value of RV x

Probability Box

- A probability box is formed by discretizing the imprecise CDF bounds into *z* intervals.



Combining Imprecise Probabilities

- Dempster Shafer Structures
- Dependency Bounds Convolutions (Williamson and Downs 1990)
- Probability Bounds Analysis (Ferson and Donald 1998)
- Many methods expanded by Ferson et al. 2003, including:
  - Enveloping
- For this method, use enveloping for combining p-boxes

P-box Imprecise Probability

- Each independent imprecise probability structure is discretized into a p-box of *z* intervals, each of 1/*z* probability mass, in order to combine multiple p-boxes

- For the developed method, p-box uncertainty is considered only in the stiffness matrix

- Because many common distributions have infinite tails, the tails must be truncated (such as at CDF=.005)

Combining P-boxes

- Because a equally discretized p-box is really a group of equally probable intervals, the p-box eigenvalue problem can be treated as a cluster of interval eigenvalue problems

Advantages of Imprecise Probability

- Allows for accurate modeling of true system and RV behavior, regardless of the level of information known

- Increased information moves $\overline{F}_i(x)$ and $\underline{F}_i(x)$ closer together, thus yielding tighter bounds with greater reliability

- Allows for engineering decision making without requiring assumptions on the CDF

- One disadvantage of Imprecise Probability is that it still requires either empirical data or expert opinion to generate the CDF bounding curves.

Dynamic Analysis

- An essential procedure to design a structure subjected to time-dependent excitations.

- In conventional dynamic analysis, the existence of any uncertainty present in the structure's geometric and mechanical properties are neglected.

Uncertainty in Dynamics

- Structure's Physical Imperfections
- Structure Modeling Inaccuracies
- Structure-Load Interaction Complexities

Conventional Modal Dynamic Analysis

- For a structure with [M] and [K] representing the global mass and stiffness matrices, and assumed modal damping ratios, ζ_n, the deterministic dynamic analysis is completed by:
  - Computing natural circular frequencies, ω_n, and corresponding mode shapes, $\{\varphi_n\}$
  - Determine maximum dynamic amplification for each mode
  - Compute modal participation factor for each mode
  - Calculate maximum modal responses
  - Combine maximum modal responses to compute maximum total response

Imprecise Probability Response Spectrum Analysis (IPRSA)

- If uncertainty is introduced into any of the structure's parameters, a new method of dynamic analysis is required capable of carrying this uncertainty throughout the dynamic analysis; the steps of IPRSA are:
  - Quantify all uncertain parameters
  - Determine uncertain bounds on natural circular frequencies and mode shapes
  - Determine uncertain bounds on dynamic amplification
  - Determine uncertain bounds on modal participation factor
  - Determine uncertain bounds on maximum modal response
  - Determine maximum bound on uncertain maximum total response

Quantification of Uncertain Parameters

- For uncertainty existing in the Modulus of Elasticity of member *i*, the uncertain parameter is defined as an uncertain coefficient times a deterministic value, or:

$$\widetilde{E}_{iq} = \left[x_{iqL}, x_{iqU}\right] * E$$

- This formulation for defining uncertainty is likewise carried through to the member's element stiffness contribution to the uncertain global stiffness matrix as:

$$[\widetilde{\mathrm{K}}_i] = \left(\left[x_{i(q_i)L}, x_{i(q_i)U}\right]\right) * [\overline{\mathrm{K}}_i]$$

- The contributions of all deterministic and uncertain members are then summed to obtain the global uncertain stiffness matrix:

$$[\widetilde{\mathrm{K}}_G] = \left(\left[x_{1(q_1)L}, x_{1(q_1)U}\right]\right) * [\overline{\mathrm{K}}_1] + \left(\left[x_{2(q_2)L}, x_{2(q_2)U}\right]\right) * [\overline{\mathrm{K}}_2] \ldots$$

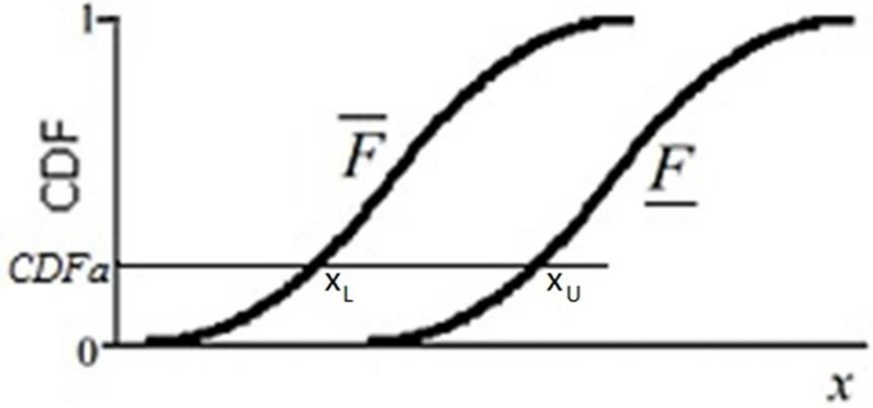
$$+\left(\left[x_{N(q_N)L}, x_{N(q_N)U}\right]\right) * [\overline{\mathrm{K}}_N]$$

Uncertain Bounds on Natural Frequencies Using Interval Monte-Carlo Frequency Analysis (IMFA)

- Bounds on natural circular frequencies can be obtained utilizing IMFA by randomly selecting CDF values for each uncertain parameter and then analyzing the interval bounds as:

$$(\boldsymbol{x}_{1rU}[\overline{\mathrm{K}}_1] + \boldsymbol{x}_{2rU}[\overline{\mathrm{K}}_2] + \dots\dots\dots + \boldsymbol{x}_{NrU}[\overline{\mathrm{K}}_N])\{u\} = (\omega_{rU}^2)[\mathrm{M}]\{u\}$$

$$(\boldsymbol{x}_{1rL}[\overline{\mathrm{K}}_1] + \boldsymbol{x}_{2rL}[\overline{\mathrm{K}}_2] + \dots\dots\dots + \boldsymbol{x}_{NrL}[\overline{\mathrm{K}}_N])\{u\} = (\omega_{rL}^2)[\mathrm{M}]\{u\}$$



Modares, Mullen & Muhanna, 2006

Uncertain Bounds on Mode Shapes

- Uncertain bounds on the mode shapes for a structure with uncertain parameters may be computed by completing a pseudo-deterministic (central) analysis to determine deterministic natural circular frequencies and mode shapes to then computed bounds on the uncertain mode shapes as:

$$\{\tilde{\varphi}_j\} = \{\varphi_j\} + \left([\Phi_j] * \left(\omega_j^{\,2} * [I] - [\Omega_j]^2\right)^{-1} * [\Phi_j]^T\right) * \left([\mathrm{M}]^{-1/2} * [\tilde{\mathrm{K}}_{G_R}] * [\mathrm{M}]^{-1/2}\right) * \{\varphi_j\}$$

where:

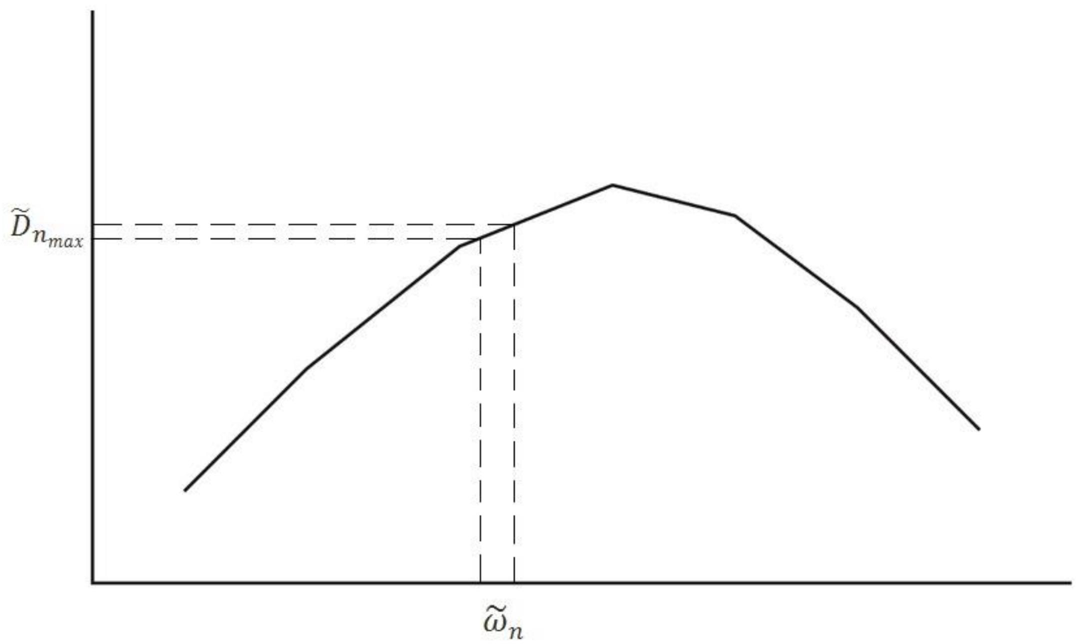
$$[\Phi_j] = \left[\varphi_1 | \varphi_2 | \dots | \varphi_{j-1} | \varphi_{j+1} | \dots | \varphi_{a-1} | \varphi_a\right]$$

and

$$[\Omega_j] = \begin{bmatrix} {\omega_1}^2 & 0 & & \cdots & 0 & 0 & & \dots & 0 & 0 \\ 0 & {\omega_2}^2 & & \cdots & 0 & 0 & & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & {\omega_{j-1}}^2 & & 0 & & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & & {\omega_{j+1}}^2 & & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & & \cdots & 0 & 0 & \cdots & & {\omega_{a-1}}^2 & 0 \\ 0 & 0 & & \cdots & 0 & 0 & \cdots & & 0 & {\omega_a}^2 \end{bmatrix}$$

Uncertain Bounds on Dynamic Amplification

- For a given response spectrum, the uncertain maximum dynamic amplification is found by evaluating the bounds on each uncertain natural circular frequency to obtain interval bounds on each uncertain maximum dynamic amplification, $\widetilde{D}_{n_{max}}$



Uncertain Modal Participation Factor

- For a structure subjected to proportional dynamic loading, or:

$$\{P(t)\} = P(t) * \{p\}$$

the uncertain modal participation factor is computed as:

$$\widetilde{\Gamma}_j = \frac{\left\{\widetilde{\varphi}_j\right\}^T * \{p\}}{\left\{\widetilde{\varphi}_j\right\}^T * [\mathrm{M}] * \left\{\widetilde{\varphi}_j\right\}}$$

Uncertain Maximum Modal Response

- The uncertain maximum modal response for the *j*th mode is computed as:

$$\left\{\tilde{u}_{j_{max}}\right\} = \tilde{D}_{j_{max}} * \tilde{\Gamma}_j * \left\{\tilde{\varphi}_j\right\}$$

- Because the uncertain modal participation factor is a function of the uncertain mode shape, the above equation must be expanded to account for dependencies as:

$$\left\{\tilde{u}_{j_{max}}\right\} = \{\tilde{D}_{j_{max}} * \frac{\{p\}^T * \left\{\varphi_j\right\} * [I_a] + \{p\}^T * \left\{\varphi_j\right\} * \left[C_j\right] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) + \cdots}{\left\{\varphi_j\right\}^T * [\mathrm{M}] * \left\{\varphi_j\right\} + \left\{\varphi_j\right\}^T * [\mathrm{M}] * \left[C_j\right] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \left\{\varphi_j\right\} + \cdots}$$

$$\frac{\{p\}^T * \left[C_j\right] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \left\{\varphi_j\right\} * [I_a] + \{p\}^T * \left[C_j\right] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \left\{\varphi_j\right\} * \left[C_j\right] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i])}{\left\{\varphi_j\right\}^T * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]^T) * \left[C_j\right] * [\mathrm{M}] * \left\{\varphi_j\right\} + \left\{\varphi_j\right\}^T * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]^T) * \left[C_j\right] * [\mathrm{M}] * \left[C_j\right] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \left\{\varphi_j\right\}} * \left\{\varphi_j\right\}$$

Modares and Mullen, 2013

Maximum Uncertain Maximum Modal Response & Maximum Uncertain Total Response

- As only the maximum of the uncertain maximum modal response is of interested for design, we evaluate:

$$\left\{\tilde{u}_{j_{max}}\right\}_{max} = \max\left(\left\{\tilde{u}_{j_{max}}\right\}\right)$$

- The total structure response is then found by combining the modal contribution of all modes. For this work, the Square Root Sum of Squares (SRSS) method is utilized to combine all maximum uncertain maximum modal responses as:

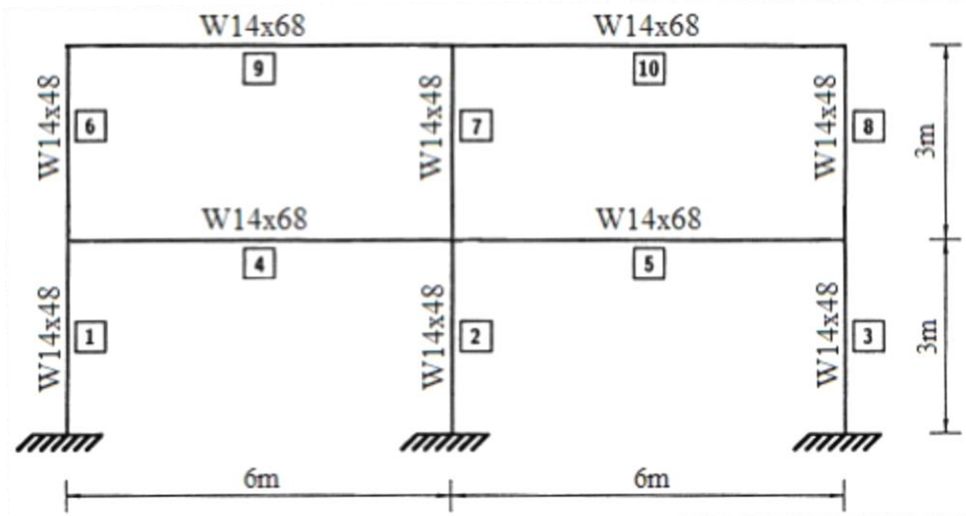
$$\left\{\tilde{U}_{max}\right\}_{max} = \sqrt{\sum_{j=1}^{a} \left\{\tilde{u}_{j_{max}}\right\}_{max}^{2}}$$

Example 1

Consider 2D, 18 DOF system shown below.

For normally distributed uncertainty only in each members' elasticity:

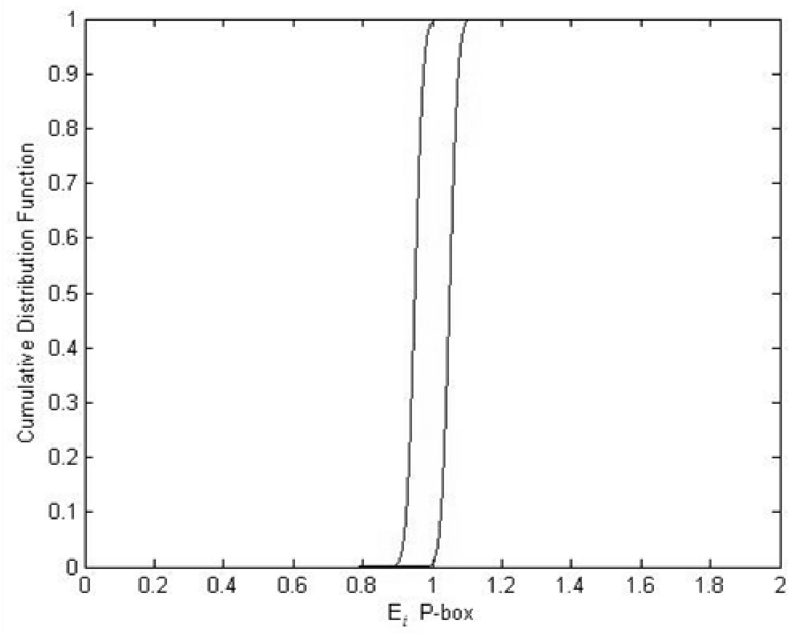
$$E_i : \tilde{\mu} = [.95, 1.05] * E, \sigma = .0194 * E$$



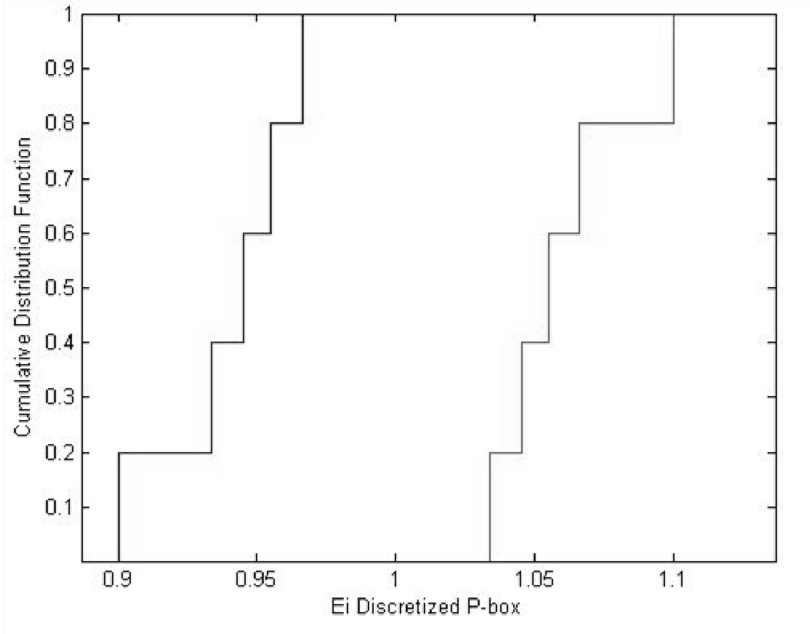
Assuming A, I & L are deterministic, solve for bounds on the system's natural frequencies and total response for first and second floors if the system is subjected to NBK Response Spectrum

P-box Discretization

- The modulus of elasticity of each member in the system is given by the bounds given on the previous slide.



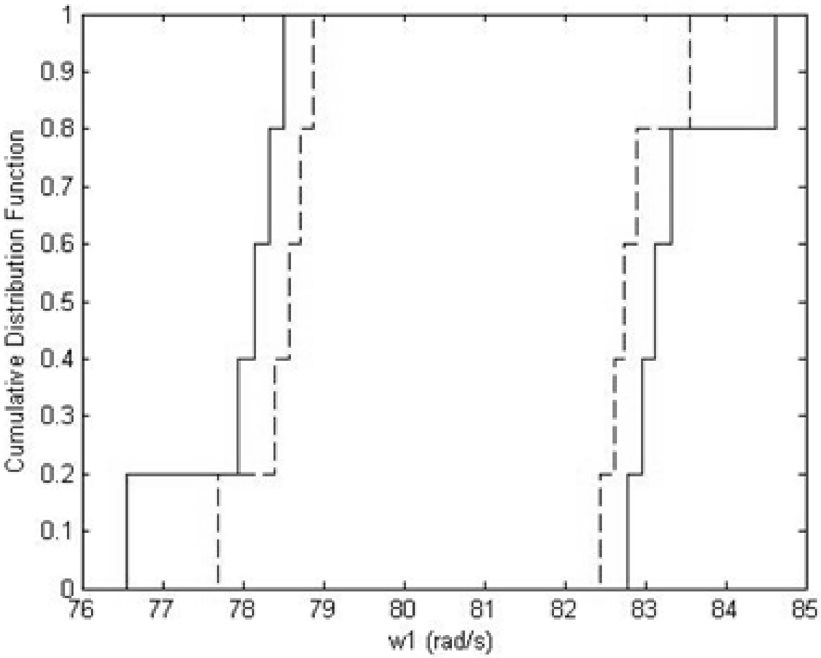
- The lower p-box shows the discretized p-box for the modulus of elasticity of each member for z=5 intervals.



Uncertain bounds on fundamental natural frequency, ω_1, Example 1

- The dashed lines in the graph represent the IMFA verification of the combinatorial results, represented by the solid lines

- IMFA results are all inner bounds of the combinatorial results

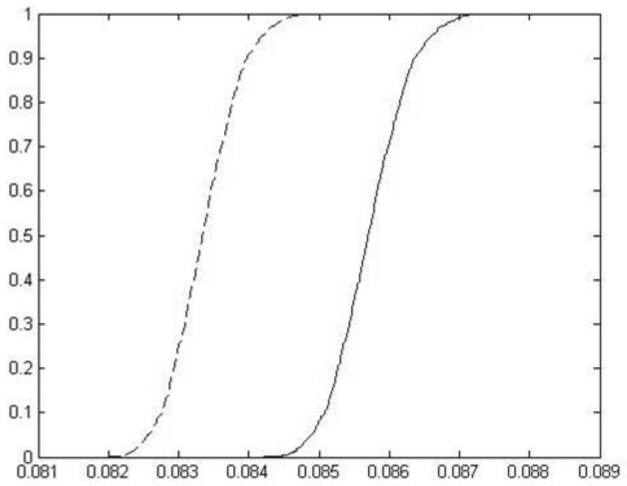


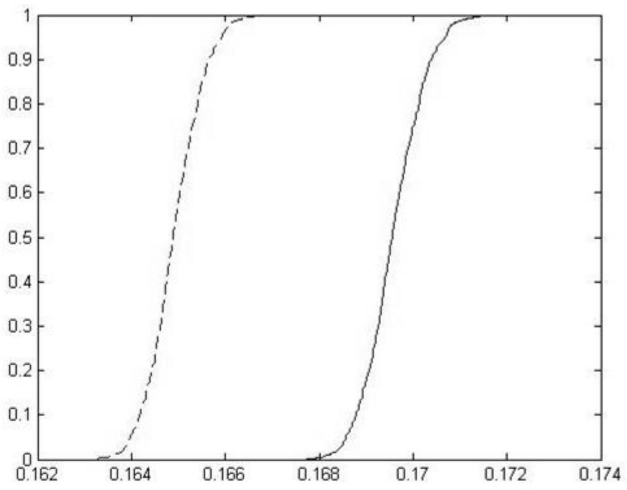
Maximum Uncertain and Pseudo-Deterministic Story Drifts

- The dashed lines in the graph represent the pseudo-deterministic (central) response for the response of each floor

- Solid lines are the maximum uncertain response of IPRSA

- Top figure is response of first floor and lower is response of second floor

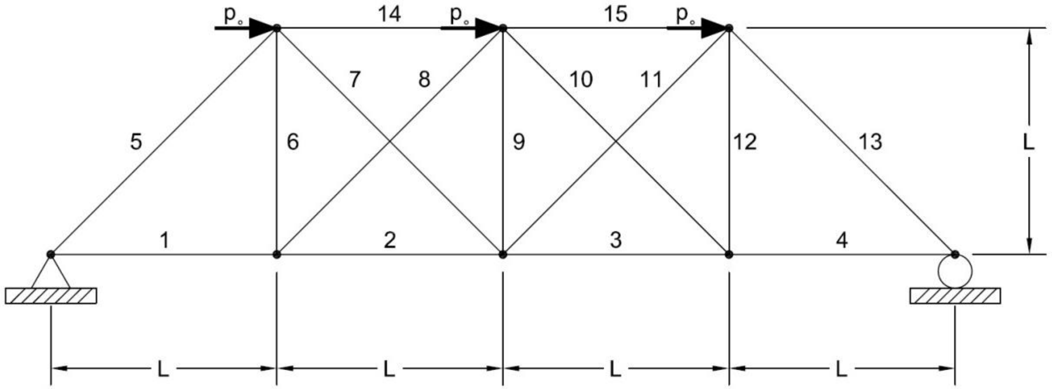




Example 2

Consider the truss system shown below.

The structure is subjected to a heavy-side step loading function.



Given all parameters are deterministic, with each member having equal cross sectional properties

Uncertainty exists only in the damping ratio for each mode defined as:

$$\zeta_n : \tilde{\mu} = [0.03, 0.07], \sigma = 0.008$$

Uncertain Maximum Modal Response

- For a heavy-side step load function, the deterministic dynamic response coordinate is computed as:

$$D_n = 1 + e^{\frac{-\zeta_n * \pi}{\sqrt{1-\zeta_n{}^2}}}$$

- Because only the maximum nodal responses are of interest, for uncertain damping ratio, only the lower bounds on the uncertain damping ratios control. Thus:

$$\widetilde{D}_{n_{max}} = 1 + e^{\frac{-\zeta_{n_{min}} * \pi}{\sqrt{1-\zeta_{n_{min}}{}^2}}}$$

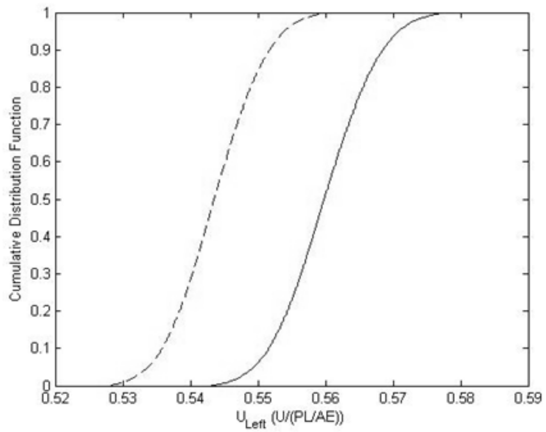
- Therefor the uncertain maximum modal response is computed as:

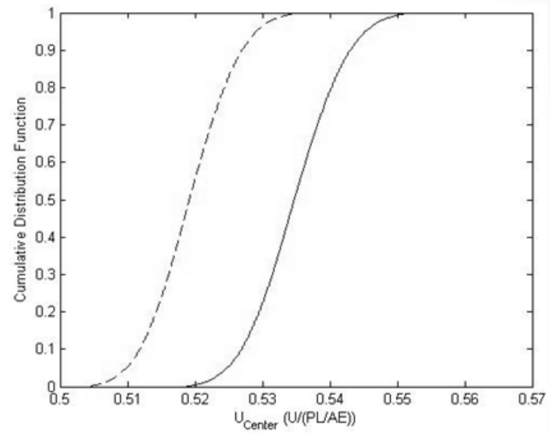
$$\left\{\tilde{u}_{j_{max}}\right\} = \widetilde{D}_{j_{max}} * \Gamma_j * \left\{\varphi_j\right\}$$

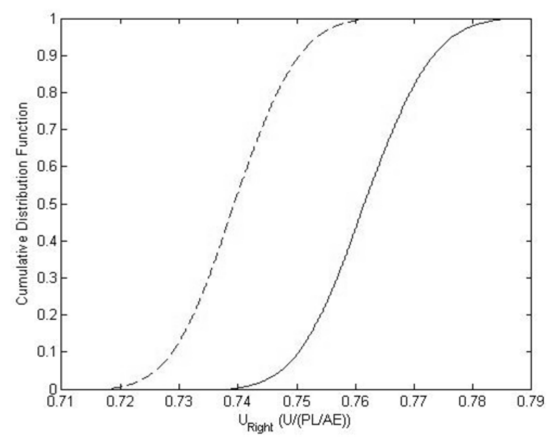
Maximum Uncertain and Pseudo-Deterministic Top Chord Nodal Responses

- The dashed lines in the graph represent the pseudo-deterministic (central) response for each top chord node of the truss

- Solid lines are the maximum uncertain response of IPRSA







Summary & Conclusions

- Imprecise Probability approach, as one of the newer polymorphic uncertainty analyses, is capable of treating uncertainties not only in the data and model, but also on their distributions.
- Using this approach, due to its set based (interval) configuration, computationally feasible schemes can be developed for uncertain system analyses.
- Application of Imprecise Probability on dynamics of a system has led to the development of computationally feasible methods that are capable of enumerating uncertainties in the PDF of input parameters.
- This method is versatile to level of uncertainty present in the system i.e. the less uncertainty (more information), the tighter the bounds on the output.

QUESTIONS