Chapter 1

# Preliminaries

## Preface

In this chapter, we collect results, notations and tools from functional analysis which will be used throughout the text. This section is, of course, not intended as a substitute for a course on functional analysis, and the reader is referred, for example, to texts such as [8] for proofs and a comprehensive introduction. The main text is based on the assumption that readers are familiar with the notions and results indicated in the present section.

## 1.1 Functional Analysis

By $\mathcal{L}(\mathcal{X},\mathcal{Y})$ we denote the set of bounded, linear operators from $\mathcal{X}$ into $\mathcal{Y}$; $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is itself a Banach space when equipped with norm

$$\|A\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} = \sup_{0\neq w\in\mathcal{X}} \frac{\|Aw\|_{\mathcal{Y}}}{\|w\|_{\mathcal{X}}} . \tag{1.1}$$

For $\mathbb{K}=\mathbb{R}$, $\mathcal{X}' \simeq \mathcal{L}(\mathcal{X},\mathbb{R})$, the space of bounded, linear functionals on $\mathcal{X}$, and likewise for $\mathbb{K}=\mathbb{C}$, $\mathcal{L}(\mathcal{X},\mathbb{C})$ shall denote the space of bounded *antilinear* functionals on $\mathcal{X}$, again denoted by $\mathcal{X}'$ (with the coefficient field implied).
co-linear

When $\mathcal{X}=\mathcal{Y}$, we write $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X},\mathcal{X})$. The subspace $\mathcal{L}_{inv}(\mathcal{X})$ of boundedly invertible operators in $\mathcal{L}(\mathcal{X})$ is a multiplicative group with multiplication given by the composition $A\circ B$ of $A, B\in\mathcal{L}(\mathcal{X})$.

We present a collection of fundamental concepts and statements in functional analysis and measure theory. For most proofs and a more in-depth discussion, we refer to [48, 41, 63, 60, 61, 77, 80] for functional analysis, [40, 60, 64] for tensor product spaces, [4, 62] for measure theory, and [~~64, 80~~] for a discussion of vector-valued integration.
97, 116 (Raymond A.) (Yosida)

### 1.1.1 Normed vector spaces

We assume the reader is familiar with the concepts of vector spaces, linear maps, norms and inner products from linear algebra. In the following, we consider only vector spaces over the field $\mathbb{R}$ of real numbers, although all statements extend to vector spaces over the complex field $\mathbb{C}$ with few or no modifications.