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ScienceDirect
Historia Mathematica 36 (2009) 213–246
HISTORIA MATHEMATICA
www.elsevier.com/locate/yhmat

On mathematical problems as historically determined artifacts: Reflections inspired by sources from ancient China

Karine Chemla *

CNRS, UMR 7596, REHSEIS, Paris, France
Université Paris 7, 3 square Bolivar, 75019 Paris, France

Available online 4 March 2009

For Menso Folkerts
On the occasion of his 65th birthday
As an expression of friendship and appreciation

Abstract

Is a mathematical problem a cultural invariant, which would invariably give rise to the same practices, independent of the social groups considered? This paper discusses evidence found in the oldest Chinese mathematical text handed down by the written tradition, the canonical work *The Nine Chapters on Mathematical Procedures* and its commentaries, to answer this question in the negative. The Canon and its commentaries bear witness to the fact that, in the tradition for which they provide evidence, mathematical problems not only were questions to be solved, but also played a key part in conducting proofs of the correctness of algorithms.

Résumé

Un problème mathématique est-il un invariant culturel, qui déclencherait les mêmes pratiques quel que soit le groupe social considéré ? Cet article analyse des témoignages prélevés dans les sources chinoises les plus anciennes à avoir été transmises par la tradition écrite, l'ouvrage canonique *Les neuf chapitres sur les procédures mathématiques* et ses commentaires, pour répondre par la négative à cette question. Ces documents attestent que, dans la tradition dont ils témoignent, les problèmes n'étaient pas seulement des énoncés à résoudre, mais qu'ils jouaient un rôle clef dans la conduite des démonstrations de correction d'algorithmes.

MSC: 01A25; 01A35; 01A85

Keywords: Mathematical problem; Ancient China; *The Nine Chapters on Mathematical Procedures*; Liu Hui; Proof of the correctness of algorithms

* Fax: +33 1 44 27 86 47.
E-mail address: chemla@univ-paris-diderot.fr.

0315-0860/$ – see front matter
doi:10.1016/j.hm.2008.11.005

Introduction

The main goal of this paper is methodological.[1] It derives from the general observation that historians have often worked under the assumption that the main components of scientific texts—problems, algorithms and so on—are essentially ahistorical objects, which can be approached as some present-day counterparts. To be clear, by this diagnosis, I do not mean that when problems reflect tax systems or civil works, historians have failed to recognize that their statements adhere to a given historical context. Rather, I claim that they have mainly viewed problems in the way we commonly use them today—or as we commonly think we do—that is, as formulations of questions that require something to be determined and call for the execution of this task. In contrast, by focusing on mathematical problems found in ancient Chinese sources, this paper aims at establishing that we cannot take for granted that we know a priori what, in the contexts in which our sources were produced or used, a problem was. Moreover, besides providing evidence to support this claim, it suggests a method that can be used to describe the nature of mathematical problems in a given historical tradition. My conclusion is that developing such descriptions should be a prerequisite to setting out to read sources of the past.

To substantiate these statements and illustrate this method, I shall concentrate on the earliest extant mathematical documents composed in China. These sources are of various types, a point that will prove essential for my argument. The book that factually is the oldest mathematical writing that has survived from ancient China, the *Book of Mathematical Procedures* (*Suanshushu* 算數書), was recently discovered in a tomb sealed ca. 186 B.C.E.[2] In contrast to this document, which was *not* handed down, the earliest book that has come down to us through the written tradition, *The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu* 九章算術), to be abbreviated as *The Nine Chapters*, was probably completed in the first century C.E. and considered a "Canon (*jing* 經)" soon thereafter.[3] Both books are for the most part composed of particular problems and algorithms solving them. What kind of texts were they? How should we read them? These are the main questions I have in view here. The outlines of the problems often echo the concrete problems that bureaucrats or merchants of that time might have confronted in their daily practice. It has hence often been assumed that mathematics in ancient China was merely practical, oriented, as it seems, toward solving concrete problems. In fact, such a hasty conclusion conceals two implicit assumptions regarding problems. The first assumption, specifically attached to sources such as the Chinese ones, is that the situation used to set a problem should be taken at face value: learning how to handle the concrete task presented by the problem would precisely be what motivated its inclusion in a text. The second assumption, much more generally held, is that the sole aim of a problem is to present a mathematical task to be executed and to provide means to do so. In this paper, I shall cast doubt on the second assumption, showing that it distorts our view of the way in which problems were actually used in ancient China. As a side result, the first assumption will also be undermined. This conclusion implies that we should be careful in the ways in which we use the evidence provided by the particular situations described in the problems

[1] A first version of this paper was presented at the conference organized by Roger Hart and Bob Richards, *The Disunity of Chinese Science*, which was held in Chicago on May 10–12, 2002. It is my pleasure to take the opportunity to thank the organizers of this meeting and the audience for their comments. A revised version was prepared at the invitation of Della Fenster, for the conference "Exploring the History of Mathematics: How Do We Know What Questions To Ask?" that was held at the University of Richmond, Richmond, VA (USA), on May 12–15, 2004. May she be thanked for her generosity and encouragement. Markus Giaquinto, the referees, and the editors of *Historia Mathematica* spent much time trying to help me formulate the argument of this paper more clearly. I owe them a huge debt of gratitude.

[2] The first critical edition was published in Peng Hao 彭浩 [2001]. Since then, several papers have suggested philological improvements, such as Guo Shirong 郭世榮 [2001], Guo Shuchun 郭書春 [2001]. Two translations into English are forthcoming: [Cullen, 2004], the first version of which was published on the internet, and [Dauben, 2008]. A new critical edition and translation into Japanese and Chinese was published recently: [張家山漢簡『算數書』研究会編 Chôka san kankan Sansûsho kenkyûkai. Research group on the Han bamboo slips from Zhangjiashan *Book of Mathematical Procedures*, 2006]. Recently, Cullen [2007] presented his rewriting of the mathematics of ancient China based on the discovery of the manuscript.

[3] I argue in favor of this dating in Chemla and Guo Shuchun [2004, 475–481]. In chapter B [Chemla and Guo Shuchun, 2004, 43–56], Guo Shuchun presents the various views on the date of completion of the book held by scholars in the past and argues that the composition of *The Nine Chapters* was completed in the first century B.C.E. In what follows, I shall regularly rely on the glossary of mathematical terms I composed to back up the translation into French of *The Nine Chapters* and the commentaries (see below) [Chemla and Guo Shuchun, 2004, 897–1035]. In it, the entry *jing* 經, "Classic, Canon," provides evidence countering the commonly held view that only in the seventh century were such books as *The Nine Chapters* considered to be "Canons." In Chemla [2001 (forthcoming), 2003b], on the basis of the extant evidence regarding *The Nine Chapters*, I discuss the kind of scripture a Canon constituted in ancient China. In addition to separate contributions, [Chemla and Guo Shuchun, 2004] contains joint work, such as the critical edition of *The Nine Chapters* and the commentaries on which this paper relies.

inserted in mathematical documents. We cannot conclude merely from their outer appearance that they were written only for practical purposes.

How can we devise a method for determining what a mathematical problem was in a given historical context? The suggestion I develop in this paper is to look for readers of the past who left evidence regarding how they read, or reacted to, our sources. This is why the Chinese case is most helpful: in the form of commentaries, it yields source material documenting early readings of some mathematical sources. This is also where the difference in type between the two Chinese documents mentioned above is essential. In relation to its status as a Canon, *The Nine Chapters* was handed down and several commentaries were composed on it. Among these commentaries, two were selected by the written tradition to be handed down together with the text of the Canon: the one completed by Liu Hui 劉徽 in 263 and the one presented to the throne by Li Chunfeng 李淳風 in 656. As we have already stressed above, *The Nine Chapters* is for the most part composed of problems and algorithms solving them. The commentaries bear on the algorithms and, less frequently, on the problems. They are placed at the end of the piece of text on which they comment or between its statements. In this paper, I shall gather evidence from Liu Hui's commentary with respect to how he used problems and also how he read those included in *The Nine Chapters*. This will provide us with source material to reconsider in a critical way the nature of a mathematical problem in ancient China and to establish that problems were submitted to a mathematical practice differing, on many points, from the one we spontaneously attach to them.

What does such evidence tell us? To start with, it shows us why the questions we raise on problems are essential and cannot be dismissed if we are to read our sources in a rational way. We shall select two examples of the difficulties which face the historian and which are illustrated by the commentary. First, *The Nine Chapters* describes a procedure for multiplying fractions after the following problem:

> (**1.19**) Suppose one has a field which is $4/7$ *bu* wide and $3/5$ *bu* long. One asks how much the field makes. 今有田廣七分步之四, 從五分步之三, 問爲田幾何。[4] [Chemla and Guo Shuchun, 2004, 170–171]

Why is it that, in the middle of his commentary on the procedure for multiplying fractions, the third century commentator Liu Hui introduced **another** problem that can be formulated as follows:

> 1 horse is worth $3/5$ *jin* of gold. If a person sells $4/7$ horse, how much does the person get?[5]

Seen from our point of view, the two problems are identical: their solution requires multiplying the two fractions by one another. Obviously, from Liu Hui's perspective, they differ. Otherwise, he would not need to change one for the other. Inquiring into their difference **as perceived by Liu Hui** should hence disclose one respect in which the practice of problems in ancient China differs from the one we would be spontaneously tempted to assume. This will be our first puzzle.

In the example above, the numerical values remain the same. The first difficulty hence regards the interpretation of the *situation* chosen to set a problem. In addition, the commentator also regularly changes the *numerical values* with which a problem is stated, without changing the situation, before he comments on the procedure attached to it. For instance, Problem 5.15 in *The Nine Chapters*, which requires determining the volume of a pyramid, reads as follows:

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4 The shape of the field is designated by the names of its dimensions: a field with only a length (north–south direction) and a width (east–west direction) is rectangular. I discuss the names of geometrical figures and their dimensions in Chemla and Guo Shuchun [2004, chapter D, 100–104]. "1.19" indicates the 19th problem of Chapter 1 of *The Nine Chapters*. The same convention is used for designating other problems in this paper. Note that at the time when *The Nine Chapters* was composed, the *bu* was between 1.38 and 1.44 m. The values of the length and width do not seem concrete.

5 Section 3 of this paper describes the text in greater detail. I refer the reader to this section to see precisely how the commentator formulates this problem. Note that procedures for multiplying fractions similar to the one in *The Nine Chapters* occur in several contexts in the *Book of Mathematical Procedures*. In some cases, the procedure is given apparently without the context of any problem (bamboo slips 6, 7 [Peng Hao 彭浩, 2001, 38, 40]). In another case, its statement follows a problem of the type Liu Hui introduces in his commentary (selling a fraction of an arrow and getting cash in return, bamboo slips 57–58 [Peng Hao 彭浩, 2001, 65]). In yet other cases, the procedure is formulated in relation to computing the area of a rectangular field and showing how it inverts procedures for dividing (bamboo slips 160–163 [Peng Hao 彭浩, 2001, 114]).

(**5.15**) Suppose one has a *yangma*, which is 5 *chi* wide, 7 *chi* long and 8 *chi* high. One asks how much the volume is. 今有陽馬,廣五尺,袤七尺,高八尺◦問積幾何◦[6] [Chemla and Guo Shuchun, 2004, 428–429]

At the beginning of his commentary, which, as always, follows the procedure, Liu Hui states:

Suppose[7] the width and the length are each 1 *chi*, and the height is 1 *chi* 假令廣袤各一尺,高一尺, . . .

Again, the same question forces itself upon us: for which purpose does the commentator need to change the numerical values here, as he also regularly does when he discusses other surfaces and solids? This will be our second puzzle.

My claim is that it is only when we are in a position to argue for an interpretation of the differences between, on the one hand, the situations and, on the other hand, the numerical values that we may think we have devised a non-naive reading of mathematical problems as used in ancient China.

This paper develops an argument for solving the two puzzles. Here are the main steps of the argument:

First, I shall show that we must discard the obvious explanation that one could be tempted to put forward, that is, that a problem in ancient China only stood for itself. We can put forward evidence showing that the commentators read a particular problem as standing for a class (*lei*) of problems. Moreover, they determined the extension of the class on the basis of an analysis of the procedure for solving the particular problem rather than simply by a variation of its numerical values. An even stronger statement can be substantiated: one can prove that these readers expected that an algorithm given in *The Nine Chapters* after a particular problem should allow solving as large a class of problems as possible [Chemla, 2003a]. In making these points, we shall thereby establish the first elements of a description of the use of problems in mathematics in ancient China.

Section 1 thus proves that our puzzles cannot be easily solved and require explanations of another kind. In Section 2, I introduce some basic information concerning the practice of proving the correctness of algorithms as carried out by commentators like Liu Hui, since this will turn out to be necessary for our argumentation.

On this basis, Section 3 concentrates on the situations used to set problems. It argues that, in fact, far from being reduced to questions to be solved, the statements of problems, and more specifically the situations with which they are formulated, were an essential component in the practice of proof as exemplified in the commentaries. Viewing the situations from this angle allows accounting for why the problem with cash was substituted for that about the area of a field.

Section 4 focuses on the numerical values given in the problems and suggests that, if we set aside cases in which a change of values aims at exposing the lack of generality of a procedure, it is only within the framework of geometry that Liu Hui changed the values in the statements of problems. In Section 4, I argue that the reason for this is that the commentators introduce material visual tools to support their proofs and that the numerical values given in the problems refer to these tools. This leads me to suggest a parallel between the role played by visual tools in the commentaries and the part devoted to the situations described by problems.

Sections 1 to 4 concentrate on how the evidence provided by the commentary allows solving the two puzzles put forward. To be sure, these puzzles could also be grasped only thanks to the commentaries. As a result, my argument establishes how the practice with problems attested to by the commentators differs from our own and how this description accounts for the difficulties presented. Even if this is to be considered as the only outcome of the paper, we would have fulfilled our aim of providing an example of a practice with problems that does not conform to our expectations. Now, the question is: can we go one step further and transfer the results established with respect to the use of problems by the commentators to *The Nine Chapters* itself, even though the earliest extant commentary on *The*

[6] The *yangma* designates a specific pyramid, with a rectangular base. Its shape is defined by the fact that its apex is above a vertex of the rectangular base. See Section 4. I opt for not translating the Chinese term, to avoid a long expression that would express the *yangma* as a kind of pyramid, whereas the term in Chinese does not link the shape to that of other geometrical solids. We do not know exactly by means of which object the Chinese term designates the pyramid. Nor do we know whether, at the time when *The Nine Chapters* was composed, the term had acquired a technical meaning or was only designating a specific object with the geometrical shape required. Note also that, at that time, 1 *chi* was between 0.23 and 0.24 m.

[7] The expression for "suppose" is not the same as in *The Nine Chapters* itself; i.e., *jinyou* has been replaced by "suppose (*jialing*)," which in the commentaries as well as in some later treatises is more frequently used to introduce a problem.

Nine Chapters was composed probably some two or three centuries after the completion of the Canon? This question is addressed somewhat briefly in Section 5.

The reason that we must proceed in this indirect way relates to the fact that commentators wrote in a style radically different from that of the Canons. More precisely, commentaries express expectations, motivations, and second-order remarks, all these elements being absent from *The Nine Chapters*, which is mainly composed of problems and algorithms. Commentaries hence allow us to grasp features of mathematical practice that are difficult to approach on the basis of the Canon itself, at least when one demands that a reading of ancient sources be based on arguments. One point must be emphasized: Unless we find new sources, there is no way to reach full certainty about whether what was established on the basis of third-century sources holds true with respect to writings composed some centuries earlier. However, this having been said, two remarks can be made.

First, relying on commentaries composed more than two centuries after the Canon to interpret the latter appears to be a less inadequate method than relying on one's personal experience of a mathematical problem. It seems to me to be more plausible that the practice of problems contemporary with the compilation of *The Nine Chapters* is related to Liu Hui's practice than that it is related to ours. However, here we are in the realm of hypothesis rather than certainty.

Second, once we restore the practice of problems to which the commentaries bear witness, we can find many hints indicating that some conclusions probably hold true for *The Nine Chapters* itself and even for the *Book of Mathematical Procedures*. Section 5 is devoted to discussing such hints. With these warnings in mind, let us turn to examining our evidence.

1. *How* does a problem stand for a class of problems?

1.1. A first description of the statement of problems

Problem 1.19 quoted above illustrates what, in general, a problem in *The Nine Chapters* looks like. It is particular in two respects. The statement of the problem refers to a particular and most often apparently concrete situation, such as, in this case, computing the area of a field. Moreover, it mentions a particular numerical value for each of the data involved—in this case, $3/5$ *bu* and $4/7$ *bu* for the data "length" and "width," respectively.[8] However, some problems are only particular in this latter respect. An example of this is Problem 1.7, one of three that precede the procedure for the addition of fractions:

> (**1.7**) Suppose one has $1/3$ (one of three parts), $2/5$ (two of five parts). One asks how much one obtains if one gathers them. 今有三分之一,五分之二,問合之得幾何。

Although numerical values are given, the fractions to be added, or in other terms, the units out of which parts are taken, are abstract.[9] All problems in *The Nine Chapters* are of one of the two types exemplified by 1.19 and 1.7.

The procedures associated with the problems in *The Nine Chapters* also show some variation. Problem 1.19 is followed by a procedure for the multiplication of parts, which amounts to $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. It is expressed in general, abstract terms, since it makes no reference to the situation in the problem, the area of a field:

> Multiplying parts
> Procedure: the denominators being multiplied by one another make the divisor. The numerators being multiplied by one

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8 The same remark holds true for the problems in the context of which the procedure amounting to the "Pythagorean theorem" is discussed. Further, in fact, there is evidence showing that at the latest in the third-century the term "field" came to designate a geometrical shape in general. See Chemla and Guo Shuchun [2004, Glossary, 992–993]. It is hence difficult to determine whether the statement of the problem in *The Nine Chapters* still uses the term with a concrete sense or already with a technical meaning. The same problem was raised above regarding the interpretation of *yangma*. More generally, the qualification of the statement of a problem as "concrete" should be manipulated with care. Vogel [1968, 124–127] describes the general format of a problem in *The Nine Chapters*. He suggests that the problems can be divided into two groups: those dealing with problems of daily life and those that can be considered as recreational problems. In my view, this opposition is anachronistic and does not fit the evidence we have from ancient China (see the following discussion). Moreover, Vogel fails to point out that there are problems formulated in abstract terms.

9 For a general discussion on the form of these problems, see Chemla [1997a]. Compare [Cullen, 2007, 17; Guo Shuchun 郭書春, 2002, 514–517] for a description of the form of problems in the *Book on mathematical procedures*.

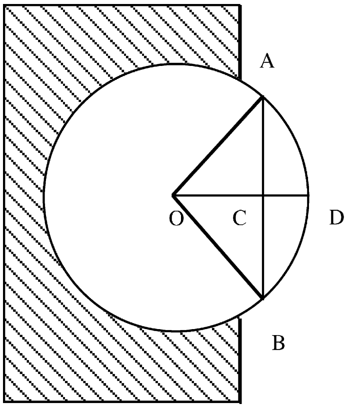

Fig. 1. The problem of the log stuck in the wall.

another make the dividend. One divides the dividend by the divisor.
乘分
術曰：母相乘爲法，子相乘爲實，實如法而一。[Chemla and Guo Shuchun, 2004, 170–171].

However, in other cases, the procedure given in *The Nine Chapters* is expressed with respect to the concrete situation and values described by the problem. An example of this is the following problem from Chapter 9, "Basis and height (*gougu*)," which is devoted to the right-angled triangle (cf. Fig. 1):

> (**9.9**) Suppose one has a log with a circular section, stuck into a wall, with dimensions unknown. If one saws it with a saw at a depth of 1 *cun* (CD), the path of the saw (AB) is 1 *chi* long. One asks how much the diameter is.
> Answer: the diameter of the log is 2 *chi* 6 *cun*.
> Procedure: half the path of the saw being multiplied by itself, one divides by the depth of 1 *cun*, and increases this (the result of the previous operation) by the depth of 1 *cun*, which gives the diameter of the log.
> 今有圓材埋在壁中，不知大小。以鐻鐻之，深一寸，鐻道長一尺。問徑幾何。
> 荅曰：材徑二尺六寸。
> 術曰：半鐻道自乘，如深寸而一，以深寸增之，即材徑。[10]

In modern terms, the procedure amounts to the formula[11] $\frac{AC^2}{CD} + CD = 2AO$. Since both the problem and the procedure are formulated in the same concrete terms, we might, for such cases at least, be tempted to assume that they are to be read as standing only for themselves. Interestingly, as we will see below, we can find evidence allowing us to determine how Liu Hui read this problem. It clearly shows that even in such cases this assumption must be discarded. Moreover, it also reveals how Liu Hui used this problem as a general statement, and not as a particular one.

1.2. The commentator Liu Hui's reading and use of problems

The piece of evidence on which we can rely to approach the commentator's reading of the latter problem comes from Liu Hui's commentary on Problems 1.35/1.36 and the procedure included in *The Nine Chapters* for the determination of the area of a circular segment. The key point is that, in this piece of commentary, Liu Hui refers to the problem of the log stuck in a wall from the Canon. The Problems 1.35 and 1.36 are similar. The second one reads as follows (cf. Fig. 2):[12]

10 [Chemla and Guo Shuchun, 2004, 714–715]. In the translation, for the sake of my argument, I have inserted references to a geometrical figure drawn by myself (Fig. 1). Needless to say, neither the figure nor the references are to be found in *The Nine Chapters*. More generally, the Canon does not refer to any visual tool.

11 In the right-angled triangle OAC, the difference of the hypotenuse and the side OC is equal to CD. Dividing AC^2 by CD yields the sum of the hypotenuse and the side OC; hence the result.

12 Again, I have drawn the figure for the sake of commenting. No such figure is to be found in the original sources. Moreover, I have added to the translation of the original text references to the figure between brackets.

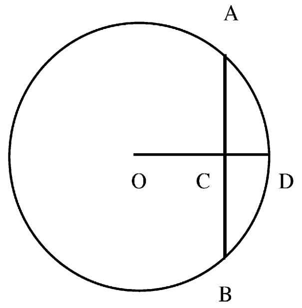

Fig. 2. The area of the circular segment.

> (**1.36**) Suppose again[13] that one has a field in the form of a circular segment, whose chord (AB) is 78 *bu* 1/2 *bu*, and whose arrow (CD) is 13 *bu* 7/9 *bu*. One asks how much the field makes. 又有弧田, 弦七十八步二分步之一, 矢十三步九分步之七。問爲田幾何。[Chemla and Guo Shuchun, 2004, 190–191]

The Nine Chapters then provides an algorithm to compute the area of this field, which amounts to the formula $(AB.CD + CD^2)/2$. In his analysis of this procedure, Liu Hui first shows that when the circular segment is half of the circle, in fact the algorithm computes the area of the half-dodecagon inscribed in the circle. Moreover, he stresses that the imprecision increases when the circular segment is smaller than the half-circle. This imprecision motivates him to establish a new procedure, which derives from tiling the circular segment with triangles and computing its area as the sum of their areas.[14] It is within this context that Liu Hui first needs to compute the diameter of the circle containing the circular segment from the two data of Problem 1.36, the chord (AB) and the arrow (CD). For this, he refers to Problem 9.9 on the log stuck in a wall as follows:

> (...) It is appropriate then to *rely on the procedure of the (problem) where one saws a log with a circular section* in the (Chapter) "Basis (*gou*) and height (*gu*)" and to look for the diameter of the corresponding (circle) by *taking the chord of the circular segment as the length of the path of the saw*, and *the arrow as the depth of the piece sawn*. Once one knows the diameter of the circle, then one can cut the circular segments in pieces. (...) 宜依句股鋸圓材之術, 以弧弦爲鋸道長, 以矢爲鋸深, 而求其徑。既知圓徑, 則弧可割分也。(My emphasis). [Chemla and Guo Shuchun, 2004, 192–193]

This piece of evidence shows that Liu Hui does not read the problem of the log stuck in a wall and the procedure attached to it as merely standing for themselves, but as expressing something more general. It thus reveals that, even in a case like Problem 9.9, where the procedure is expressed with reference to the concrete situation of a log stuck in a wall and particular values, the third-century commentator reads its meaning as exceeding this particular case. This holds true in the entire commentary, in which one can find other pieces of evidence confirming this conclusion, as we will see below.

1.3. The procedure defines the extension of the class meant by the problem

Another interesting example of the generality the commentator attaches to a problem and the corresponding procedure is the following one:

> (**6.18**) Suppose that 5 persons share 5 coins (units of cash) in such a way that what the two superior persons obtain is equal to what the three inferior persons obtain.[15] One asks how much each obtains. 今有五人分五錢, 令上二人所得與下三人等, 問各得幾何。[Chemla and Guo Shuchun, 2004, 526–529]

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[13] This is the usual beginning for problems after the first one, when a mathematical question is dealt with through a sequence of problems.

[14] This reasoning implies covering an area with an infinite number of tiles and it has been the topic of much discussion in the literature. It falls outside the topic of this paper. For a concise exposition, compare [Li Yan and Du Shiran, 1987, 68–69].

[15] As the commentator makes clear, it is assumed that the five persons have unequal ranks defined by the integers 5 to 1 and that the share they obtain depends on their rank. We may feel that the statement of the problem is incomplete, but this would mean that we project our own expectations of how a problem should be formulated. Actual description of how problems recorded in historical documents were formulated should replace this anachronistic approach. However, dealing with this topic would exceed the scope of this paper.

The procedure following the answers to the problem is also expressed with reference to the particular situation and values mentioned in the statement. However, in this case, through the analysis of the procedure that he then develops, Liu Hui brings to light that the procedure is not general. To be more precise, the procedure adequately solves the problem, but cannot solve all similar problems, because it uses features specific to the situation described in Problem 6.18. Note that this is the only case when this happens for a procedure given in *The Nine Chapters* and that the commentator immediately exposes the lack of generality. This reaction betrays his expectation that a problem stands not only for itself, but also for a class (*lei*). Liu Hui reacts to this situation in several steps.

First, his analysis determines criteria that enable him to know to which problems the procedure of *The Nine Chapters* can be successfully applied.[16] In a sense, Liu Hui inquires into the class of problems for which the particular problem stands, and he does so through an examination of the *procedure*.

Second, he formulates another problem similar to that of the Canon, as follows:

> Suppose[17] 7 persons share 7 coins and they want to do this in such a way that (what) the two superior persons (obtain) is equal to (what) the five inferior persons (obtain). 假令七人分七錢,欲令上二人與下五人等。[Chemla and Guo Shuchun, 2004, 528–529]

The criteria previously put forward immediately show why the procedure of *The Nine Chapters* does not apply to this problem.[18] Despite the appearances, on the basis of the procedure stated, the two problems do not belong to the same class. In other terms, the category of problems for which a problem stands is not determined by a variation of its numerical values, but rather by the procedure provided to solve it.

Third, Liu Hui suggests modifying the procedure in such a way that it solves all similar problems. Seen from another angle, Liu Hui aims at stating a procedure for which all similar problems belong to the same class. We see that the commentary on a procedure analyzes it in such a way as to inquire into the extension of its validity and modifies it to extend the class of problems that can be solved by it (and for which a particular problem stands). As a result, on the basis of the sections of Liu Hui's commentary examined so far, and in fact of others, we can thus state that, in his view, a problem stands for a class (*lei*) of problems that is determined on the basis of the procedure described after it. It is not so much the similarity of structure between the situations described by different problems that allow considering them as sharing the same category, but, most importantly, the fact that they are solved by the same procedure.[19] We hence reach the conclusion that far from being only the sequence of operations allowing a given problem to be solved, the procedure is read beyond the specific context within which it is formulated, and, further, it even determines the scope of generality of a given problem. We shall come back to this issue in Section 3.

In the fourth step, the one we are most interested in here, Liu Hui suggests an entirely different, more general procedure for dealing with Problem 6.18. In fact, the commentator does this simply by suggesting "to imitate the procedure" given in the Canon for the next problem, 6.19. This problem reads as follows:

[16] These criteria are as follows: the number of inferior persons must exceed the number of superior ones by only 1; moreover, the sum of the coefficients attached to the superiors (5 and 4) must be greater than that attached to the inferiors $(3, 2, 1)$. I do not enter into any detail here, referring the interested reader to Chemla [2003a].

[17] Here too, the expression *jinyou* for "suppose" in *The Nine Chapters* has been replaced by *jialing*. In this paper, I do not discuss the numerical values chosen to set a problem. However, clearly they call for comment. The figures used in Problem 6.18 in *The Nine Chapters* are the simplest possible with which the mathematical question can be formulated. In his commentary, Liu Hui introduces values that are the simplest possible to make his point. Also, the figures occurring in Problem 1.19 (3, 4, 5, 7; see above) are probably chosen on purpose. Compare my introduction to Chapter 9 in Chemla and Guo Shuchun [2004, 663–665, 684–689]. Further research is needed in this respect.

[18] There are three more persons among the inferiors than among the superiors. Moreover, the sum of the coefficients attached to the superiors $(7 + 6)$ is smaller than the sum of those attached to the inferiors $(5 + 4 + 3 + 2 + 1)$.

[19] On this point, I refer the reader to Chemla [1997a], where I analyze how Liu Hui uses the term *lei* "class, category" with respect to problems. As rightly stressed by C. Cullen, in the earliest known theoretical discussion on the modes and methods of inquiry in mathematics and cosmography, i.e., in the opening sections of *The Gnomon of the Zhou* (*Zhou bi*), which he dates to the beginning of the common era, the concept and practice of "categories" in mathematics are central [Cullen, 1996, 74–75, 177]. In my glossary of mathematical terms, I discuss more generally the various uses of the term *lei* in the commentaries on *The Nine Chapters* and in philosophical texts of antiquity [Chemla and Guo Shuchun, 2004, 48–949]. It is interesting that the *Book of Mathematical Procedures* also attests to the use of the term *lei* in the description of mathematical procedures as early as the second century B.C.E. (see bamboo slip 21 [Peng Hao 彭浩, 2001, 45]).

(**6.19**) Suppose that a bamboo has 9 internodes[20] and that the 3 inferior internodes have a capacity of 4 *sheng* whereas the 4 superior internodes have a capacity of 3 *sheng*. One asks, if one wants that between two (neighboring) inner internodes capacities be uniformly distributed,[21] how much they each contain. 今有竹九節,下三節容四升,上四節容三升。問中間二節欲均容,各多少。

Again, the procedure described after the statement of Problem 6.19 refers to the particular situation and values displayed in the statement. However, despite the differences on both counts between Problems 6.18 and 6.19, Liu Hui *directly* imitates the procedure solving 6.19 (that is, transfers it step by step) to solve Problem 6.18, and, beyond, the problems that now belong to the same class. As a result, the new procedure shapes Problem 6.18 as standing for a much larger class.

In this case, as in the case of the commentary following Problem 1.36 quoted above, the same phenomenon recurs: the procedure circulates from one context to the other, disregarding the change in situation and in numerical values.[22] What is particularly noteworthy, however, is *how* Liu Hui does so in both contexts. Let us explain on the example of the commentary on the circular segment (Problem 1.36) what we mean by the "circulation" of a problem. Liu Hui does not feel the need to express a more abstract statement or procedure that would capture the "essence" of Problem 9.9 and could be applied to similar cases such as Problem 1.36. On the contrary, he directly makes use of the procedure given after 9.9, with its own terms, in the context of 1.36, by establishing a term-to-term correspondence, "*taking the chord of the circular segment as the length of the path of the saw*, and *the arrow as the depth of the piece sawn*." This seems to indicate that the situation described in Problem 9.9 can be directly put into play in other concretely different situations. The particular appears to be used to state the general in the most straightforward way possible.

Further, to describe a more general procedure for problems of the same class as 6.18, Liu Hui imitates the procedure for 6.19 within the context of the most singular of all problems (6.18) and not that of the more "generic" one, which he introduced as a counterexample. More importantly, for all problems such as 6.18, he could simplify the procedure given for 6.19 to make it fit certain specific features that these problems all share—for problems like 6.18, in contrast to the bamboo problem and its middle internodes, there are no persons who do not belong to either the group of inferiors or that of superiors. Instead, Liu Hui prefers to keep the procedure with the higher generality that characterizes it. The conclusion of the previous paragraph can be stated in a stronger way: the most particular of all paradigms is used to formulate the most general of all algorithms and consequently it now stands for a much wider class.

A remark concerning the use of problems in the context of commentaries is here in order and will prove useful below. Liu Hui uses the procedure solving Problem 9.9 in a commentary in which he describes a new procedure for computing the area of the circular segment. *At the same time* as he shapes the procedure, he shows why it is *correct*. Such a concern for the correctness of algorithms drives the greatest part of the commentaries, which systematically establish that the procedures in *The Nine Chapters* are correct. Problems play a key role in achieving this goal. In the context examined, the use of Problem 9.9 is signaled by the verb "to look for *qiu* 求."[23] More generally, this term signals the use of a problem in a proof. The commentary on the area of a circular segment illustrates the following use of a problem in a proof: The task of establishing the new procedure is divided into subtasks, which are identified with problems known—to start with, Problem 9.9. On the one hand, using the procedure that *The Nine Chapters* gives to solve Problem 9.9 yields the first segment of the procedure now sought for. On the other hand, since the correctness of the procedure solving 9.9 has already been established, the commentator can rely on the fact that it yields the magnitude needed at this point of the reasoning, that is, the diameter of the circle. In the terms Liu Hui uses to speak of the proof of the correctness of a procedure, the "meaning *yi* 意" of the result of the first segment of the procedure has been ascertained. More generally, let us stress the fact that, within the proof of the correctness of an algorithm he is shaping, Liu Hui uses problems and procedures solving them to determine step by step the "meaning" of the whole sequence of operations, that is, to determine that the procedure he establishes yields the area of a circular segment.

[20] The trunks of bamboos have nodes. The Chinese term refers here to the space between two nodes and considers the "capacity" of the volume thus formed. The problem hence deals with nine terms of an arithmetic progression—the capacities of the successive spaces between the nodes. The sums of the first three terms and of the last four ones, respectively, are given and it is asked to determine all the terms.

[21] As in the previous problem, the capacities of the cavities form an arithmetic progression.

[22] Compare the analysis of transfer between situations in Volkov [1992, 1994].

[23] Note that the verb occurs in the statement of problems and tasks in the *Book of Mathematical Procedures*; see bamboo slips 160–163 [Peng Hao 彭浩, 2001, 114].

1.4. A similar way of reading and using problems in later sources

In ancient China, this way of using procedures stated in one context directly in another, illustrated above for the use of the procedure solving Problem 9.9 for dealing with the area of a circular segment, was not specific to Liu Hui. In fact, one can find a similar piece of evidence four centuries later, in a seventh-century commentary on the *Mathematical Canon Continuing the Ancients* (*Qigu suanjing*), written by Wang Xiaotong in the first half of the seventh century.[24] The commentary relates the first problem of the book, which is devoted to astronomical matters, to a problem dealing with a dog pursuing a rabbit, indicating that the latter problem is included in *The Nine Chapters*.[25] In this case, too, the latter problem is not reformulated in astronomical terms, nor is a third and abstract description of it introduced as a middle term, to allow the result concerning the dog and the rabbit to be applied in astronomy. In exactly the same way as described above, although *The Nine Chapters* presents the problem within a particular concrete context, the first reader that we can observe, namely the seventh-century commentator, reads it as exemplifying a set of problems sharing a similar structure and solved by the same algorithm. Furthermore, as in the previous example, the commentator feels free to make the problem and procedure, which apparently do not relate to astronomy, "circulate" as such into a different, astronomical context. This seems to indicate that there was an ongoing tradition in ancient China that did not mind discussing general mathematical procedures in the particular terms of the problems in which they had been formulated, although the questions discussed exceeded the case illustrated by the particular situation. The above evidence from the third and the seventh centuries leads to the same conclusions. This indicates that it would not be farfetched to assume that this was also the way in which the authors of *The Nine Chapters* conceived of the problems that they included in the Canon. This seems all the more reasonable because, as we have indicated above, except for one case (6.18), all procedures following the statement of problems are general.[26] We shall come back to this issue in greater detail in Section 5.

Even though it is perhaps less striking in comparison with our own uses of problems, let us stress that, in fact, the same pieces of evidence show that these conclusions hold true with respect to the numerical values. Although Liu Hui regularly comments on a given problem and procedure on the basis of particular numerical values, he understands the meaning of his discussion as extending beyond this particular set and as, in fact, general. Again, in this respect, the commentator thus proves to discuss the general in terms of a particular [Chemla, 1997a].

The problems of logs stuck in walls and dogs pursuing rabbits that can be found in *The Nine Chapters* may be perceived as recreational by some readers of today, because of the terms in which they are cast. The evidence examined proves that things are not so simple. The historian is thus warned against the assumption that the category of "mathematical problem" remained invariant in time. Such a historical reconstruction guards us against mistaking a problem as merely particular or practical, when Chinese scholars read it as general and meaningful beyond its own context, or mistaking it as merely recreational when it was put to use in concrete situations.

Now that we have seen that, in Liu Hui's practice, a problem did practically stand for a category of problems, and how it did so, we have discarded the simple solution that could have accounted for our puzzles. We are hence left with the question: Why is it that, within the context of his commentary on the procedure for "multiplying parts," or on that for the volume of the *yangma*, Liu Hui feels it necessary to substitute one situation for another, or one set of values for another, although both the original and the substituted problem seem to us to share the same category? One may even

[24] Volume 2 of Qian Baocong [1963] contains a critical edition of Wang Xiaotong's *Qigu suanjing*. The problem and commentary mentioned are to be found in Qian Baocong [1963, 2, 495–496]. Eberhard (Bréard) [1997] discusses this example. Part of her discussion and her translation are published in Bréard [1999, 41–43, 333–336]. For a further study concerning transmissions of problems of this type, see Bréard [2002].

[25] Although *The Nine Chapters* contains similar problems, the extant editions do not contain precisely the one quoted. Since we are only interested here in how a problem dealing with a given situation is used in another context, this textual problem can be left aside.

[26] This fact is only one feature among the many hints indicating that generality was a key epistemological value that inspired the composition of *The Nine Chapters*. Another hint is provided by how the "chapters *zhang* 章" are composed. They each embody a part of mathematics that derives from a unique procedure, and in correlation with this fact, their text is organized around the generality of the procedure placed at the beginning, which by derivation commands the whole chapter. Alexei Volkov stresses this fact in his translation of the title of the Canon as *Computational Procedures for Nine Categories* [*of Mathematical Problems*] [Volkov, 1986, 2001]. Compare the translations of the title as *Nine Chapters on the Mathematical Art* [Needham and Wang Ling, 1959, 19], *Nine Books on Arithmetical Techniques* [Vogel, 1968], *Nine Chapters of Calculation* [Wang Ling, 1956, 15], *Mathematical Methods in a Nine-Fold Categorization* [Cullen, 2002, 784], and *Mathematical Procedures under Nine Headings* [Cullen, 2004, 1]. Among the meanings that may be intended by the title, the term *zhang* designates a division in a writing or a stage in a process of development, as well as, more generally, a distinction. Whatever the interpretation of the title may be, the division of the book into nine chapters manifests the influence of the value of generality.

say that after examining the evidence presented so far, our puzzles look even more intriguing. Elaborating a solution for these puzzles will compel us to enter more deeply into the practice of mathematical problems as exemplified in Liu Hui's commentary.

2. Proving the correctness of procedures in order to find out general formal strategies in mathematics

The interpretation of Liu Hui's commentary on the procedure for "multiplying parts" or computing the volume of the *yangma* requires that we recall some basic information regarding the mathematical practices linked to the exegesis of such a Canon as *The Nine Chapters*.

As has been recalled above, after virtually every procedure given by *The Nine Chapters* for solving a problem or a set of problems, the commentators systematically establish its correctness. However, the way in which they deal with the issue of correctness manifests a specific practice of proof, which can be linked to the context of exegesis within which it develops.[27] I shall sketch its main characteristics, since it will prove useful for solving our puzzles.

To this end, I shall first evoke Liu Hui's commentary on the algorithm given by *The Nine Chapters* for adding up fractions, which appears to be a pivotal section in his text.[28]

2.1. Proving the correctness of the procedure for the addition of fractions

In Chapter 1, where the arithmetic of fractions is dealt with, the Canon presents three problems similar to Problem 1.7, after which it offers the following general and abstract algorithm for adding up fractions, equivalent to $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$:

> Gathering parts
> Procedure: The denominators multiply the numerators that do not correspond to them; one adds up and takes this as the dividend (*shi*). The denominators being multiplied by one another make the divisor (*fa*). One divides the dividend by the divisor. (...)
> 合分
> 術曰：母互乘子，幷以爲實。母相乘爲法。實如法而一。不滿法者，以法命之。[Chemla and Guo Shuchun, 2004, pp. 156–161].

Let us outline how, in his commentary on this section of the Canon, Liu Hui establishes the correctness of this procedure.

The Canon and the commentaries approach the object "fraction," or in Chinese terms: "parts," in two ways. The expression for m/n used in *The Nine Chapters*, "*m* of *n* **parts**" (*n fen zhi m n* 分之 *m*), gives the fraction as being composed of "parts." This dimension is the one emphasized in what I call the "material" approach to fractions. The expression also displays a numerator and a denominator ("***m*** of ***n*** parts"), which are the basis for what I designate as the "numerical" approach to fractions. These two approaches to fractions appear to have been combined by the mathematicians of ancient China. On the one hand, the problem asking us to add up fractions requires gathering various disparate parts together to form a single quantity, which must hence be evaluated ($\frac{a}{b} + \frac{c}{d} = ?$). On the other hand, the algorithm prescribes computations on numerators and denominators—the numerical dimension of the fractions involved—to yield a value as the result of a division ($\frac{ad+cb}{bd}$—where $ad + cb$ is the dividend and bd the divisor). Establishing the correctness of the algorithm requires proof that the value obtained ($\frac{ad+cb}{bd}$) measures the quantity formed by bringing together the various parts ($\frac{a}{b} + \frac{c}{d}$).

In his commentary on the simplification of fractions, a topic dealt with immediately before the addition of fractions, Liu Hui had approached the fractions as entities to be manipulated by the procedure concerned, i.e., as a pair consisting of a numerator and a denominator, and he had stressed the potential variability of their expression: one can multiply, or

[27] The question of the relationship between the exegesis of a Canon, on the one hand, and Liu Hui's or Li Chunfeng's specific practice of proof, on the other hand, is addressed in Chemla [2001 (forthcoming)]. In Chapter A of Chemla and Guo Shuchun [2004, 27–39], I discuss the fundamental operations that the commentaries put into play when proving the correctness of procedures.

[28] I discuss in detail this annotation of Liu Hui's commentary in Chemla [1997b]. For further argumentation regarding what is stated in this section, the reader is referred to this paper.

divide, the numerator and the denominator by the same number, he had stated, without changing the quantity meant. In this context, to divide (e.g., 2/4 becoming 1/2) is to "simplify *yue* 約" the fraction. The opposed operation (e.g., 2/4 becoming 4/8), which Liu Hui had then introduced and called by opposition "to complicate *fan* 繁," is needed only for the sake of proving the correctness of algorithms dealing with fractions.

At the beginning of his commentary on the procedure for "gathering parts," Liu Hui, then, considers the counterparts of these operations with respect to the fractions regarded as parts: "simplified" fractions correspond to "coarser parts" (halves instead of fourths), "complicated" ones to "finer" parts (eighths instead of fourths). At this level, Liu Hui again stresses the invariability of the quantity, beyond possible variations in the way of composing it (using halves, fourths, or eighths).

Now to prove that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

the commentator shows that the algorithm carries out the following program: by multiplication of both the numerator and the denominator, it refines the disparate parts to make them have the same size (a/b becoming ad/bd and c/d becoming cb/bd, so that all parts are bdths). Quoting the Canon, Liu Hui expounds the actual "meaning" of each step, in terms of both parts and numerators/denominators, making clear how they combine to fulfill the program outlined.[29] When "the denominators are multiplied by one another"—an operation that, in the course of the proof, Liu Hui names "to equalize"—this computes the denominator common to all fractions (bd) and defines the size that the different parts can have in common: when parts have a common size, the fractions can be added. Moreover, when "the denominators multiply the numerators that do not correspond to them," to yield ad and bc, the numerators, he says, are made homogeneous with the denominators to which they correspond. The overall operation is a "complication." It has been previously shown to be valid and it ensures that the "original quantities are not lost." Here, too, in passing, Liu Hui confers a name on this set of operations: "to homogenize."

"Equalizing" the denominators and "homogenizing" the numerators, the algorithm thus actually yields a correct measure of the quantity formed by gathering the various fractions.

2.2. *The correctness of algorithms and the search for fundamental operations*

We, contemporary readers, may read Liu Hui's commentary on the procedure for "gathering parts" as establishing the correctness of an algorithm. But what were the aims pursued by the commentator when writing it? They are highlighted by the following part of this commentary, which continues with highly abstract and philosophical considerations, concluded by a key declaration: "Multiply to disaggregate them, simplify to assemble them, homogenize and equalize to make them communicate, how could those not be the key points of computations/mathematics (*suan*)?" [Chemla and Guo Shuchun, 2004, 158–159].[30]

Opaque as it may seem, this declaration is essential. It clearly shows that something else is at stake in the previous proof besides establishing the correctness of a procedure. The proof had exhibited operations at play in the algorithm: multiplying numerators and denominators to disaggregate the parts they represent, dividing them to assemble the parts into coarser parts, equalizing denominators, homogenizing numerators. These operations constitute the topic of the subsequent considerations: exhibiting them appears to be one motivation for carrying out the proof. Moreover, all these operations were introduced in relation to fractions, for which they referred to precise operations on numerators and denominators. However—and this is the point difficult to understand—Liu Hui's concluding declaration indicates

29 As above, in the discussion of the commentary on Problem 9.9, the term "meaning" refers to what the operations carry out, with respect to the situation in which they are applied. One term in Liu Hui's terminology can be interpreted to correspond to this concept: "meaning, intention *yi* 意." See my glossary in Chemla and Guo Shuchun [2004, 1018–1022].

30 In the English translation of *The Nine Chapters* by Shen Kangshen et al., various critical parts of the work are not translated accurately and their importance is hence overlooked. For the passage considered, compare the translation by Shen Kangshen et al.: "Multiplying [the denominators] means fine division and reducing means rough division; the rules of homogenizing and uniformizing are used to get a common denominator. Are they not the key rules of arithmetic?" [Shen Kangshen et al., 1999, 72]. In addition to inaccuracies (such as, for instance, translating "assemble" by "rough division"), the theoretical import and generality of the statement, which is one of the most important of the commentary, is completely missed. The consequences should appear clearly with what is explained below, in this section and in the following one.

that their relevance far exceeds this limited context, since they are now listed among the "key points of mathematics." How can we interpret this claim?

In fact, when we read Liu Hui's commentary as a whole, we observe that these operations recur in several other proofs that the commentator formulates to establish the correctness of other procedures described in the Canon [Chemla, 1997b]. Let us allude to an example.[31] Chapter 8 of the Canon is devoted to solving systems of simultaneous linear equations. If we represent a system by the equations

$$bx + ay = e,$$
$$dx + cy = f,$$

the fundamental algorithm given in *The Nine Chapters* amounts to transforming them into

$$bdx + ady = ed,$$
$$bdx + bcy = bf.$$

Then, by subtraction of the two equations, one eliminates x and determines the value of y, after which the value of x is easily obtained. When Liu Hui accounts for the correctness of the algorithm, he brings to light that one can multiply and divide the coefficients of an equation by the same number without altering the relationship it expresses between the unknowns. Moreover, he points out that the algorithm "equalizes" the values of the coefficients of x, whereas it "homogenizes" the values of the other coefficients. This is how all the operations included in the key declaration quoted above occur again in the proof of the correctness of another algorithm. The same holds true for other cases, as we shall shortly see.

This fact explains why Liu Hui's declaration can be so general and why he makes a statement, the validity of which goes beyond the context of fractions. However, if we compare the two situations alluded to, in which Liu Hui identifies "equalization," clearly in the context of equations we do not have an "equalization of denominators," since what is "equalized" is the coefficients of x. In other words, what is common between the two contexts is *not* the concrete meaning that "equalization" takes, although in each context the specific concrete meaning of "equalization" is what is required for the proof to work.

The declaration invites us to find something else that is common to the various contexts in which the operations identified occur and which would justify its validity. This leads us to note that in each proof in which they occur, the terms designating the operations have in fact two meanings. Let us explain this point for the term "equalization."

In the context of adding up fractions, equalization was interpreted as the operation equalizing denominators. In that of equations, it was interpreted as the operation that made the coefficients of x equal. Similarly, in all the other contexts in which "equalizing" occurs, it can be interpreted in terms of its effect with respect to the particular situation to which it is being applied. This material effect constitutes the first meaning of "equalization," one that changes according to the context. However, the fact that the operation recurs in different contexts reveals that the term takes a second meaning, a formal one that is common to all contexts: the term "equalizing" points to *how* the algorithms work. All algorithms for which the proof of correctness highlights that "equalizing" and "homogenizing" are at play proceed by "equalizing" some quantities while "homogenizing" others. The second meaning of the two terms captures and expresses the strategy followed by the procedure. And the parallel between the proofs discloses that, in fact, the algorithms follow the same formal strategy of equalizing and homogenizing. Even though the concrete meanings of equalizing and homogenizing vary according to the contexts in which they are at play, formally, they operate in the same way. These remarks reveal a key feature shared by the proofs: They bring to light that the same fundamental algorithm underlies various procedures. It is on the basis of the actual reasons accounting for the correctness of the algorithms that, through the proofs, a concealed formal connection between them is unveiled. This conclusion

[31] For the purpose of clarity the example has been simplified in its detail: cf. [Chemla, 2000]. In Section 3, I shall describe with greater detail how "equalizing" and "homogenizing" occur in another context.

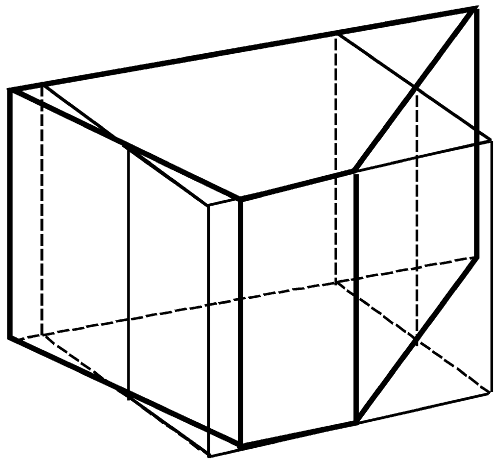

Fig. 3. The transformation of the trapezoid prism.

reveals that while proving the correctness of an algorithm, the commentator concentrates on formal dimensions in the procedure.[32]

This concern relates to one of the reasons for Liu Hui to carry out proofs, i.e., bringing to light such fundamental formal strategies common to the various procedures provided by *The Nine Chapters*.[33] Such key algorithms, such as “equalizing/homogenizing,” allow a reduction of the variety of procedures of the Canon, uncovering a *small number of strategies* systematically used in *designing all its procedures*. Multiplying or dividing all numbers in an adequate set, as well as equalizing and homogenizing, appear to underlie many of the algorithms of the Canon in domains that for us belong to arithmetic or algebra. This is why, when they first occur, in the context of the procedure for “gathering parts,” Liu Hui immediately stresses their importance. Thus his declaration appears to gather together the most fundamental algorithms underlying the procedures of *The Nine Chapters*, those procedures being brought to light by the proofs contained in his commentary.

2.3. *A fundamental operation in geometry*

The same motivation of disclosing fundamental operations common to various algorithms appears to permeate the proofs that Liu Hui develops in the context of geometry. This can be deduced from the fact that the proofs establishing the correctness of the most important algorithms related to geometrical shapes all bring to light that these algorithms use the same formal strategy, which Liu Hui captures with the expression “one uses the excess to fill up the void.”[34] Let us illustrate this point with the example of the trapezoid prism (see Fig. 3—again no figure is to be found for this problem either in the commentary or in *The Nine Chapters*). This solid is the first one considered in Chapter 5, in which most problems of that type are gathered.

The procedure given in *The Nine Chapters* to compute the volume of solids of this shape reads as follows:

> Procedure: one adds the upper and lower widths and halves this (the result). One multiplies this (the result) by the height or the depth. Again, one multiplies this (the result) by the length, hence the *chi* of the volume 術曰：并上下廣而半之。以高若深乘之, 又以袤乘之, 即積 尺。[Chemla and Guo Shuchun, 2004, 410–413][35]

[32] We could capture this point in the way in which the commentary unfolds. Several hints indicate that the authors of *The Nine Chapters* also considered procedures from a similar perspective. However, dealing with this issue would exceed the scope of this article.

[33] Such a motivation appears to be driving later commentators as well [Chemla, 2001 (forthcoming), 2003b].

[34] This point was first discussed in great detail in Wu Wenjun 吳文俊 [1982]. See further developments in Volkov [1994].

[35] For reasons that will be presented in Section 4, I have drawn the solid in a position different from the one the procedure refers to. What correspond to the “lower” and “upper width” are shown on the figure as the front and rear width. The following argument is not affected by this rotation.

To establish its correctness Liu Hui writes:

> In this procedure, the reason that “one adds the upper and lower widths and halves this” is that if *one uses the excess to fill up the void*, this yields the average width. “Multiplying this (the result) by the height or the depth” yields the erected surface of a front. The reason that “again, one multiplies this, (the result) by the length” is that it yields the volume corresponding to the solid; this is why this makes “the *chi* of the volume” 此術“并上下廣而半之”者,以盈補虛,得中平之廣。“以高若深乘之”,得一頭之立冪。“又以袤乘之”者,得立實之積,故爲“積尺”。(My emphasis) [Chemla and Guo Shuchun, 2004, 412–413]

Given the position in which I have drawn the trapezoid prism, what Liu Hui calls a front is represented in the upper and lower planes. The first steps of the procedure are interpreted as computing the area of a face, by means of its transformation into a rectangle. With the expression “one uses the excess to fill up the void,” Liu Hui indicates the concrete transformation of the solid that is at the basis of the proof (it is illustrated in Fig. 3) and that allows him to interpret the “meaning” of the successive steps of the procedure. In addition, most importantly for us here, the commentator refers to this transformation by the same sentence that he uses to designate other different concrete transformations that make the proof work in other geometrical contexts. In each context, the actual transformations differ. However, the recurrence of the same formula to refer to them reveals that viewed from a certain angle, they are formally the same.

This conclusion confirms what we have seen with “multiplying,” “dividing,” “equalizing,” and “homogenizing”: the proof again appears as a means for bringing to light formal patterns that are common to various algorithms despite the apparent difference between them that derives from the fact that they prescribe different computations.[36]

How does the proof fulfill the function of revealing such formal patterns? The example of the procedure for “multiplying parts,” which we will analyze in the following section, highlights that the problems play a key part in enabling the proof to fulfill this function and, in this case, disclose the hidden action of “equalizing” and “homogenizing” in its process. It is from this perspective that we can now go back to our first puzzle and offer a solution for it.

3. The situation in the statement of a problem as a condition for exhibiting formal strategies

As has already been mentioned, Problem 1.19, after which *The Nine Chapters* states the procedure for “multiplying parts,” requires computing the area of a rectangular field, 3/5 *bu* long and 4/7 *bu* wide. However, the procedure itself is formulated without reference to any concrete situation. Liu Hui’s commentary on the procedure provides key evidence for understanding the part played by problems for proofs to fulfill the function brought to light in the previous section. Let us analyze it in greater detail.

3.1. The first proof of the correctness of the procedure for multiplying fractions

In the first part of his commentary on this procedure, Liu Hui develops abstract reasoning to account for its correctness. This argument shows one way in which multiplication and division are at play in the design of the procedure. Its opening section can be translated as follows:

> In each of the cases when a dividend does not fill up a divisor,[37] they hence have the name of numerator and denominator.[38] If there are parts (i.e., a fraction), and if, when expanding the corresponding dividend by multiplication, then,

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[36] For the sake of clarity, we opposed the first set of general operations, presented in Liu Hui’s key declaration, to the transformation “one uses the excess to fill up the void,” which occurs only in relation to geometrical shapes. Most probably, this type of transformation was conceived of as one of the general patterns with which the operations identified in the key declaration could take shape.

[37] This technical expression refers to the case when the dividend is smaller than the divisor. Note that “dividend” designates the content of a position on the calculating surface, and not a determined number—such a way of employing terms corresponds to the assignment of variables, whose use for the description of algorithms is characteristic in ancient China, in contrast to other ancient traditions. As a result, in what follows, the word “dividend” will designate different values, depending on the operations that have been applied to the value put in the position at each step of the computation. I have respected this technical use of terms in the translation.

[38] In such cases, the result of the division is the fraction whose numerator and denominator are respectively the dividend and the divisor. “Numerator” and “denominator” refer to the numbers as constituting a fraction; “dividend” and “divisor” refer to them as the terms of the operation yielding the fraction. The commentary alternately uses the two sets of terms, with the greatest precision.

> correlatively, it (i.e., the dividend produced by the multiplication) fills up the divisor, the (division) hence only yields an integer.[39] If, furthermore, one multiplies something by the numerator, the denominator must consequently divide (the product) in return (*baochu*). "Dividing in return," this is dividing the dividend by the divisor. 凡實不滿法者而有母,子之名。若有分,以乘其實而長之,則亦滿法,乃爲 全耳。又以子有所乘,故母當報除。報除者,實如法而一也。(…)。[Chemla and Guo Shuchun, 2004, 170–171, 768, footnote 176]

Before translating the end of the argument, let us explain the meaning of some of the technical expressions. The expression of "dividing in return" (*baochu*) is particularly important to note. The commentator introduces it here for the first time in the whole text (Canon and commentaries). Seen as an operation, as Liu Hui makes it clear, it consists in a division. However, the qualification "in return" adds something to the prescription of a division: it makes explicit the **reason** for dividing. Using "dividing in return" means that it was needed, for some reason, earlier in the procedure, to carry out a multiplication, which was superfluous with regard to the sought-for result: this division compensates for the earlier multiplication, deleting its effect. This general idea clearly makes sense in the passage of Liu Hui's commentary translated above. The use of the technical term indicates that multiplying by the **numerator** is to be interpreted as follows: instead of multiplying "something" by a fraction a/b, one multiplies the "something" by its numerator a. Multiplying the fraction a/b by b, one obtains the numerator a. Having multiplied the "something" by the numerator a, instead of a/b, one has multiplied by a value that was b times what was desired. Consequently, one has to "divide in return" by that with which one had superfluously multiplied, that is, by b. Multiplying something by a/b is hence shown to be the same as multiplying by a alone, and dividing the result by b.

Most importantly for our purpose, "dividing in return" is one of the qualifications of division that one can find in *The Nine Chapters* itself too. This fact calls for two remarks. First, the expression clearly adheres to the sphere of justification, since the very prescription of the division indicates a reason for carrying it out. In other words, such hints prove that *The Nine Chapters* does refer to arguments supporting the correctness of algorithms in some specific ways. Second, we can find the expression **both** in *The Nine Chapters* and in the commentaries. This fact points to continuities between the two, which will be useful in our argumentation below.

The continuation of Liu Hui's first proof of the correctness of the procedure for multiplying fractions interprets the meaning of the procedure as follows:

> Now, "the numerators are multiplied by one another,"[40] the denominators must hence each divide in return.
> 今子相乘則母各當報除。[Chemla and Guo Shuchun, 2004, 170–171]

For each of the numerators of the fractions to be multiplied, the argument developed above applies. If one multiplies the numerators instead of multiplying the fractions, one must divide the product by each of the denominators. The last sentence of Liu Hui's first proof transforms the sequence of operations just established to carry out the multiplication of fractions (multiplication of the numerators, division by a denominator, division by the other denominator) into the procedure given in the Canon, as follows:

> Consequently, one makes the "denominators multiply one another" and one divides altogether (*lianchu*) (by their product).
> 因令分母相乘而連除也。[Chemla and Guo Shuchun, 2004, 170–171][41]

[39] If the division of a by b yields a/b, then, if a is multiplied by kb, the new dividend akb divided by b yields the integer ak.

[40] This is a quotation from the procedure of the Canon.

[41] Shen Kangshen et al. [1999, 82] translate the whole passage as follows: "When the dividend is smaller than the divisor, then [one] gets a [proper] fraction. When the numerator is multiplied by the dividend, the product may be larger than the denominator (divisor), thus yielding an integer. If the numerator is multiplied, the product should be divided by the denominator. Since the product of [all the] numerators is taken as dividend, it should be divided by the product of [all the] denominators, i.e., take the continued product of the denominators as divisor." I do not see what, in the second sentence, "multiplying the numerator by the dividend" may mean. Moreover, there is no word for "numerator" in the text at this point. Later, "if the numerator is multiplied" does not conform to the syntax of the Chinese and the translation expresses a meaning unclear to me, leaving aside (as below) the key word "divide in return." In correlation with this, the following sentence is not translated. The transformation of two divisions into a division by the product is hidden by the addition, in many places, of the term "product," which is not in the original text. The mathematical explanation given in footnote 1 (p. 83) does not seem to me to fit with the meaning of the text. It takes the commentary as distinguishing various cases in the multiplication and fails to read the argument that Liu Hui makes in it.

Transforming a sequence of two divisions into a unique division by the product of the divisors is a valid transformation because the results of division are exact. Liu Hui proves to be aware of the link between the two facts, that is, between the validity of the transformation and the exactness of the results of division [Chemla, 1997/1998]. As above, the formulation "to divide altogether" prescribes a division in a way that indicates the reason why the division prescribed is to be carried out in this particular way. This expression recurs regularly in Liu Hui's commentary. The qualification adheres to the sphere of justification. In contrast to the "division in return," however, the expression *lianchu* never occurs in *The Nine Chapters* itself.

This sentence concludes Liu Hui's first proof of the correctness of the procedure. One can see *how* the proof discloses that multiplication and division, two of the fundamental operations listed in Liu Hui's declaration, are put into play, as opposed operations, for designing the procedure as it stands. For our purpose, it is interesting to note that this proof of the correctness of the algorithm develops independently of the framework of the problems in the context of which the "procedure for multiplying parts" was formulated, that is, the problems about rectangular fields. The arguments only make use of general properties of dividend and divisor, numerator and denominator. Moreover, they bring into play the properties of multiplication and division with respect to each other.

However, Liu Hui does not end here his commentary on this procedure. He goes on to develop a second proof, which highlights how, seen from another angle, this procedure also puts into play "equalizing" and "homogenizing." This is where we go back to the first puzzle that we have presented.

3.2. The second proof of the correctness of the procedure for multiplying fractions

The sentence linking the two parts of the commentary is important for our purpose. Liu Hui states:

> If here one makes use of the formulation of a field with length and width, it is difficult to have (the procedure) **be understood with a greater generality** *nan yi guang yu* 此田有廣從, 難以廣諭。[Chemla and Guo Shuchun, 2004, 170–171]

In fact, this sentence justifies why the commentator discards Problem 1.19 in the context of which the procedure for "multiplying parts" is described in *The Nine Chapters* and why he introduces, immediately thereafter, another problem equivalent to the original one: 1 horse is worth 3/5 *jin* of gold. If a person sells 4/7 horse, how much does the person get? Let us analyze more closely the assertion introducing the alternative proof that the commentator develops in the sequel. Since it is a key assertion for my argument, I shall insist on it and distinguish all the facts that it reveals.

First, Liu Hui's above statement establishes a link between the context of a problem—here the problem provided by *The Nine Chapters*, i.e., that of a rectangular field—and the aim of **understanding** the procedure. It appears here that a given problem can prevent one from gaining a **wider** understanding of the procedure. Second, this seems to indicate that Liu Hui also perceives the preceding passage as contributing to an understanding of the procedure. This link between establishing the correctness of a procedure and understanding it is not fortuitous. It will recur again in the passage of the commentary following the above statement.

Now, the change of problem is justified by the attempt to gain a more general understanding of the procedure. Strikingly enough, if we bring all these facts together, the statement indicates a possible link between the context of a problem and the proof of the correctness of a procedure. Again, this link will be confirmed in what follows. In fact, Liu Hui next introduces a sequence of three new problems, all formulated with respect to a single kind of situation, which make it possible to develop the second proof of the correctness of the procedure conforming to the practice of demonstration sketched in Section 2. Let us stress right at the outset the essential consequence that can be derived from both the above statement and the analysis developed below: problems appear here not to be reduced to questions that require a solution—as we would readily assume—but they also play a part in proofs.

Observing Liu Hui's next development should hence allow us to progress on two fronts. It should provide evidence showing *how* problems intervene in proving the correctness of a procedure. And this is where the difference between the problem with the field and those with the persons, the horses, and the gold should become apparent. Moreover, it should help us grasp how Liu Hui conceives of "understanding a procedure."

Let us first indicate, in modern terms, the main idea of the second proof of the correctness of the procedure. It can be represented by the following sequence of expressions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bc} \cdot \frac{bc}{bd} = \frac{ac}{bd}.$$

The second proof brings to light that the computations of ac and bd, with which the procedure for "multiplying parts" begins, have in fact the meaning of "homogenizations." This meaning can be seized if the "equalizing" of bc, which is essential for disclosing the pattern, is revealed. These are the key points of the second argument. We shall examine below how Liu Hui puts them into play in providing his second proof.

In this, an essential part will be played by an operation the importance of which was already stressed with respect to proofs: interpreting the "meaning" (*yi*) of the computations prescribed by an algorithm, that is, expressing the "meaning" of their results—their "semantics"—in terms of the situation described by the problem. This remark enables us to infer why the first problem cannot be used here. The situation of the field with length and width is unsuitable from a semantic point of view: it does not offer possibilities of interpretation rich enough for the "meaning" of the computation of *bc* to be expressed in a natural way. As a consequence, the problem of computing the area of a rectangular field does not allow bringing to light the "equalization" that underlies the procedure for "multiplying parts." The scheme of "equalizing" and "homogenizing" cannot be unfolded in this context.

In contrast, the situation in which the sequence of three new problems is formulated, with the three components constituted by the quantities of gold, persons, and horses, offers richer possibilities for interpreting the effects of operations. It is semantically rich enough for disclosing that the scheme of "homogenizing" and "equalizing" underlies the procedure for "multiplying parts." Consequently, the new situation allows revealing in another way how the procedure for "multiplying parts" relates to the fundamental operations identified by Liu Hui and discussed in Section 2. This is what is at stake in the second proof.

The key point here is that the interest in bringing to light how equalization is at play in "multiplying parts" belongs **only** to the sphere of proving. The equalization plays no role in the actual computation of the result. This is why the problem in the context of which the algorithm is described differs from the problems in relation to which the proof is carried out. A link is thereby established between bringing to light a formal strategy accounting for the correctness of the procedure and interpreting the "intentions," the "meanings" (*yi*) of the operations in the field of interpretation offered by the situation of a problem. The example of the multiplication of fractions, in which the description of the algorithm and the development of its proof require a sequence of different problems, reveals the essential part played by problems in establishing the correctness of algorithms. Note, however, that the numerical values are common to all problems: Liu Hui introduced the new problem mentioned above in such a way that the task to be carried out is still to multiply $3/5$ by $4/7$. As a consequence, the computations leading to a solution of the problem included in *The Nine Chapters* (3 times 4, 5 times 7, dividing the former result by the latter) are an actual subset of the operations involved in proving the correctness of the procedure for "multiplying parts" (which furthermore includes computing 4 times 5).

With these observations in mind, let us examine in detail how Liu Hui uses problems to conduct his second proof.

3.3. The part played by mathematical problems in a proof

Liu Hui's second proof of the correctness of the procedure for "multiplying parts" consists in articulating, in an adequate way, a sequence of equivalent problems that are transformed one into the other. What would be done in modern mathematics by formal computations is here carried out by interpreting the results of operations with respect to a succession of problems.

In the alternative proof, Liu Hui's first step consists in formulating a first problem, the solution of which requires use of the last operation of the procedure for "multiplying parts," i.e., dividing 12 (ac) by 35 (bd). It reads as follows:

> Suppose that one asks: 20 (bc) horses are worth 12 (ac) *jin* of gold. If one sells 20 (bc) horses and if 35 (bd) persons share this [the gain], how much does a person get? Answer: $12/35$ *jin*. To solve it, one must follow the procedure for "directly sharing (*jingfen* 經分)"[42] and take 12 *jin* of gold as dividend and 35 persons as divisor. 設有問者曰, 馬二十匹, 直金

[42] This procedure, which is described in *The Nine Chapters* just before the procedure for "multiplying parts," covers all possible cases of division involving integers and fractions [Chemla, 1992a]. Note that division is dealt with before multiplication and that, in correlation with this fact, the

十二斤。今賣馬二十匹，三十五人分之，人得幾何。荅曰：三十五分斤之十二。其爲 之也，當如經分術，以十二斤金爲實，三十五人爲法。[Chemla and Guo Shuchun, 2004, 170–171]

The fact that the procedure given to solve this problem is correct was established by Liu Hui in his commentary on the preceding section of *The Nine Chapters*, devoted to “sharing parts.” This problem is immediately followed by another problem, presented as a transformation of the former one:

Suppose that, modifying (the problem), one says: 5 (b) horses are worth 3 (a) *jin* of gold. If one sells 4 (c) horses and if 7 (d) persons share this (the gain), how much does a person get? Answer: $12/35$ *jin*. 設更言馬五匹，直金三斤。今賣四匹，七人分之，人得幾 何。荅曰：人得三十五分斤之十二。[Chemla and Guo Shuchun, 2004, 170–171]

The key point here is that the procedure for solving this second problem is first described in terms of “homogenizing”:

To do it, one has to **homogenize** these quantities of gold (ac) and persons (bd). They then all conform to the first problem and this is solved by the (procedure for) “directly sharing.” 其爲之也，當齊其金，人之數，皆合初問入於經分矣。[Chemla and Guo Shuchun, 2004, 170–171][43]

The use of the term “homogenize” implies that, in parallel, an equalization is carried out. Only the operation of equalization can confer the meaning of “homogenization” on the other operations. Liu Hui will make this point explicitly in one of the subsequent sentences of his proof. “Homogenizing” quantities of gold and persons goes along with “equalizing” quantities of horses. This is why, as Liu Hui states, the homogenizations prescribed lead to the “first problem”: indeed, they yield the two following statements, in which we recognize the first problem:

20 (bc) horses are worth 12 (ac) *jin* of gold.
If one sells 20 (bc) horses and if 35 (bd) persons share [the gain], how much does a person get?

This transformation of the second problem into the first one involves computing ac and bd, which amount to the first part of the procedure for “multiplying parts.” The transformation is clearly correct: it does not alter the meaning of the relationship between the values considered. The procedure solving the first problem can then conclude the solution of the second one. Now, if one considers the whole procedure that correctly solves the second problem, through transforming it into the first one, one realizes that the procedure for “multiplying parts” is embedded in it. What makes the difference between the two is that the procedure for “multiplying parts” does not prescribe the computation of bc. However, if we pay closer attention to the way in which Liu Hui formulates the transformation of the second problem into the first one, we observe that there, too, he evokes the computation of bc only in an allusive way, by referring to the computation of ac and bd as “homogenizations,” and by stating that the second problem is reduced to the first one. In fact, the sequence of operations that is concretely given to solve the second problem corresponds exactly to the procedure given by *The Nine Chapters* for “multiplying parts.” To recapitulate, Liu Hui describes the procedure that correctly solves the second problem as being the list of operations that constitute the procedure for “multiplying parts.” While stressing in his following statements the identity between the two procedures step by step, most importantly, Liu Hui transfers the interpretation in terms of homogenization and equalization into the procedure for “multiplying parts.” He writes:

If this is so, “multiplying the numerators by one another to make the dividend (ac)” is **like homogenizing** the corresponding gold. “Multiplying the denominators by one another to make the divisor (bd)” is **like homogenizing** the corresponding persons. **Equalizing** the corresponding denominators makes 20 (bc). **But the fact that the horses be equalized plays**

exegete makes use of the procedure for division in his commentary on multiplication. To make my argument clearer, I have inserted in the translation modern symbolic expressions between parentheses.

43 Shen Kangshen et al. translate this sentence as follows: “By the Homogenization and Uniformization Rule one can get the same answer as by the rule of division.” [Shen Kangshen et al., 1999, 83], which does not fit with the Chinese. Again, the argument developed by Liu Hui cannot be understood from the translation.

no role. One only wants to find the homogenized (quantities) and this is all. 然則分子相 乘爲實者，猶齊其金也；母相乘爲法者，猶齊其人也。同其母爲二十，馬無事於同，但欲求齊而已。[Chemla and Guo Shuchun, 2004, 170–171]

At this point, it is established that, on the one hand, the procedure for “multiplying parts” solves the second problem correctly and, on the other hand, the procedure involves only homogenizations. Liu Hui devotes the following statement to considering the ambiguous status of “equalization.” The computation of *bc* as the equal quantity in the statement of the first problem is essential to ascertain that *ac* and *bd* correctly correspond to each other in a pattern similar to that of the first problem. In other words, it is essential for the proof. But this computation is useless in obtaining the result: once one knows the reason why *ac* and *bd* correspond to each other, it suffices to divide one by the other to yield the solution. This explains why Liu Hui emphasizes that the “equalization” plays no part in the procedure itself, except for allowing an interpretation of its first steps as “homogenizations.” In turn, the interpretation of these steps as “homogenizations” is what lies at the basis of the correctness of the procedure.

The key point to note here is that exhibiting the homogenizations and equalizations can only be done within the framework of the new situation with persons, gold, and horses, in which the equalization can be interpreted and thereby brought to light, and not within the framework of computing the area of a rectangular field.

Liu Hui’s last step is to show that, as far as the question raised is concerned, the second problem is equivalent to a third problem, itself strictly identical to Problem 1.19. The idea is the following: we now know that the procedure for “multiplying parts” correctly solves the second problem. We want to show that this procedure correctly solves 1.19. The step to be taken is to show that the two problems amount to the same. Liu Hui establishes the equivalence between the two by showing the equivalence between the second problem and one strictly identical to Problem 1.19, but formulated in terms of horses, gold and persons. He states:

> Moreover, that 5 horses are worth 3 *jin* of gold, these are *lü* in integers. If one expresses them in parts, then this makes 1 horse worth 3/5 *jin* of gold. That 7 persons sell 4 horses is that 1 person sells 4/7 horse. 又，馬五匹，直金三斤，完全之率；分而言之，则 爲一匹直金五分斤之三。七人賣四馬，一人賣七分馬之四。[Chemla and Guo Shuchun, 2004, 170–171]

Qualifying as *lü* the data in each of the two statements of the quote designates the ability of the pairs to be possibly multiplied or divided by a common number, without altering the meaning of the relationship [Guo Shuchun 郭書春, 1984b]. If we use this property, as Liu Hui suggests, to transform the outline of the second problem into an equivalent problem, we obtain the third problem, the one with which we have formulated our first puzzle:

> 1 horse is worth 3/5 (a/b) *jin* of gold. If one sells 4/7 (c/d) horse, how much does the person get?

This quotation brings to a close the passage of Liu Hui’s commentary on “multiplying parts” that we needed to analyze to solve our first puzzle and to account for why Liu Hui substituted the third problem for Problem 1.19. We now return to the two questions we have raised in our analysis above.

First, how can we qualify the understanding of the procedure for “multiplying parts” yielded by Liu Hui’s commentary here? In fact, as was the case for the procedure for “gathering parts,” Liu Hui proves the correctness of the procedure by bringing to light how the “procedure of homogenizing and equalizing” underlies it.

As we have emphasized above, the “meanings” of “equalizing” and “homogenizing” differ in the two contexts. In the procedure for “gathering parts,” “equalizing” meant equalizing the denominators. Here, “equalizing” refers to the fact that a denominator and a numerator are made equal so that the procedure can fulfill its task. Even though the concrete meanings differ, the procedures use the same strategy formally: in both contexts, it is by making some quantities equal and then making other quantities homogeneous that one can obtain the solution. This is what the proofs disclose. This remark indicates yet again the sense in which, in Lu Hui’s eyes, the fundamental operations he listed can be deemed fundamental for mathematics.

Each proof sheds a different light on the procedure. Through highlighting how the first operations prescribed by the procedure for “multiplying parts” amount to homogenizing some quantities, the second proof discloses a new meaning for the algorithm, that is, another way of conceiving its formal strategy.[44] The new problems introduced appear to be

[44] The commentators use another term to designate this second type of “meaning.” To avoid confusion with the first one, I transcribe it as *yi’* 義. See the corresponding entry in my glossary in Chemla and Guo Shuchun [2004, 1022–1023].

an essential condition for formulating the new understanding, based on the pattern of homogenizing and equalizing. This is because they allow the disclosure of the part played by "equalization" by providing the means to interpret the meaning of the operation. It is in this way that we can interpret Liu Hui's introductory statement, where he claims that he needs to discard the problem of the area of the rectangular field in order to "have (the procedure) be understood with a greater generality."

The second proof requires introducing a procedure—the procedure of homogenization and equalization—in which the procedure for "multiplying parts" is embedded. This brings us back to the remark I made in Section 1, on the commentary Liu Hui devoted to Problem 1.36 and the procedure for the area of the circular segment: proofs of the correctness of procedures regularly include establishing, or formulating, new procedures. For multiplying fractions, the interpretation of the operations of a procedure developed for the sake of the proof required the introduction of a new problem. In the case of the commentary on the topic of Problem 1.36, in order to formulate his new procedure, Liu Hui introduces new elements in the figure of the circular segment: the circle from which it derives, and then triangles tiling its surface. These elements are in fact the topics of other problems in *The Nine Chapters* and are thus associated with procedures that the commentator uses to establish his new procedure for the area of the circular segment. In particular, Liu Hui employs these new elements to express the "meaning" of the steps of the new procedure. We hence see here a parallel emerging between the uses of, on the one hand, geometrical figures and, on the other hand, problems. In what follows, we shall analyze further this parallel between problems and geometrical elements in a figure.

The distinction between the procedure required by the proof and the procedure for solving a problem is important for explaining the substitution of one problem for the other in the case of the multiplication of fractions. The difference between the new problem and the one described in *The Nine Chapters* now appears precisely to lie in the fact that the procedure developed within the context of the proof can be "interpreted" only with respect to the new problem. The key point is that the operation of "equalization," which is part of the procedure needed by the proof, is not needed by the procedure that computes the result. In correlation with this, the new problem, and not the one in *The Nine Chapters*, allows an interpretation of the "meaning *yi*" of the procedure. These remarks highlight *how* problems play a key part in Liu Hui's practice of proof. They account for the fact that problems that appear to be the same to us are in fact different for him.

This is how I suggest that our first puzzle can be solved. Its solution reveals that, in the ancient Chinese mathematical practice to which these documents bear witness, problems did not boil down to being statements requiring a solution, but were also used as providing a situation in which the semantics of the operations used by an algorithm could be formulated in order to establish its correctness. This second function becomes apparent when the problem fulfilling the first function cannot fulfill the second. This aspect appears to be an essential component of the practice of proof as carried out by the commentators. What is most interesting, furthermore, as the case discussed here shows, is that the interpretation of the operations with respect to the situation described to formulate a problem is used to carry out the proof and thereby disclose the latter's most formal dimensions.

Second, the previous development allows us to go back to the question of determining the class of problems for which a particular problem stands. In Section 1, we have shown that a problem stands for a class of problems sharing
Ex. the same category. But this conclusion seemed to be contradicted by the need to replace the problem about the area of a rectangular field by a problem with horses, gold, and persons, both problems sharing exactly the same particular values: the two problems looked identical to each other. In fact, there is no contradiction if we recall that the category (*lei*) associated to a problem is defined by examining the procedure attached to it. With respect to the procedure for "multiplying parts," these problems belong to the same category, but, with respect to the procedure developed for the sake of the proof, they can no longer be substituted one for the other. However, what is essential is that all problems similar to the one with horses, gold, and persons, in which the procedure of the proof, i.e., equalization and homogenization, can be unfolded share the same category. Consequently, even though the proof is discussed in terms of horse, gold, and person, with particular values, it is meant to be read as general.[45] This conclusion holds true for a

[45] In his unpublished dissertation, Wang Ling [1956, 179–181] discusses the commentary on the multiplication of fractions in relation to the problem with horses, people, and cash. He mentioned neither the change of problem, nor the pattern of "homogenizing"/"equalizing." Moreover, his entire analysis develops in terms of "proportion," which does not fit with the concepts in the text. However, his conclusion of the use of the context of a problem is worth quoting (p. 181): "Evidently, he [Liu Hui] chose these specific numbers, 3, 4, 5, 7 and problems about horses, money, and persons in a representative sense. Thus we have here an example demonstrating all the logical steps to prove a rule."

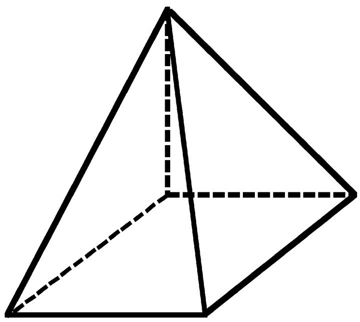

Fig. 4. The pyramid called *yangma*.

procedure as well as for a proof: here as above, when the problem of the piece sawn was used within the context of the field with the shape of a circular segment, the general is discussed in terms of the particular. This conclusion cannot be underestimated. Proofs developed by the commentators were regularly written down in ways that conflict with our expectations, since they seem to be formulated only with respect to particular cases. As a result, they were interpreted as lacking generality. Such an interpretation is in my view anachronistic, because it reads our sources in ways that project our modes of interpretation onto documents that require other readings. I think that the previous discussion establishes why they must be read as expressing general arguments.

The same conclusion can be drawn not only for the situation in which a problem is formulated, but also for the particular values it involves. Proofs relying on problems with particular values are also meant to be general in this sense: the example of the pyramid with a square basis (*yangma*) provides a clear illustration. We shall now use this example to establish this point.

4. The values in the statement of a problem: a parallel between problems and visual tools

To analyze the relevance of the numerical values used in the statement of a problem, we shall use the same method as we did in the previous paragraph, in which we focused on changes in the situations used to formulate a problem. Here, we shall examine cases in which the commentator modifies the numerical values of a problem presented in *The Nine Chapters*, before he begins commenting on a given topic.

In our discussion of Problem 6.18 in Section 1, we have seen that Liu Hui changed the values of the problem to expose the lack of generality of the procedure given by *The Nine Chapters*. This use employs problems as counterexamples, in a fashion quite common today. The change of numerical values in Problem 5.15, which deals with the volume of the pyramid *yangma*, already mentioned in the Introduction, is more difficult to account for. We will now examine the context in which it is carried out and the way it is used.

4.1. The proofs for the singular and the general cases

Chapter 5 of *The Nine Chapters* deals with the determination of the volumes of various kinds of solids. The type of pyramid called *yangma* 陽馬, in Problem 5.15, is represented in Fig. 4. Also in this case, there is no diagram in the sources. However, Liu Hui's commentary does refer to a visual tool of a different kind, namely "blocks *qi* 棊." This is the only type of visual tool the commentators use for space geometry, whereas for plane geometry they use "figures *tu* 圖." In the diagrams below, I have tried to picture only what the text says about blocks without adding more modern ways of dealing with visual aids. In particular, I have not added letters by which one could refer to the vertices of a block. Moreover, I have drawn some of the diagrams in such a way as to give a sense that the actual visual tools used were composed of solid pieces assembled in space.

The procedure stated by *The Nine Chapters* for solving Problem 5.15 prescribes multiplying all three dimensions ("width *guang* 廣," "length *zong* 從," and "height *gao* 高") by each other and then dividing the result by 3 to yield the

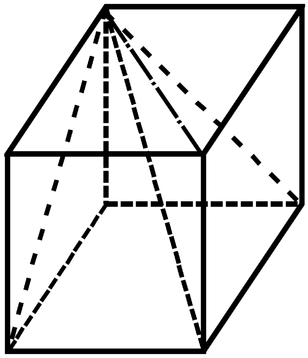

Fig. 5. Three *yangma* 陽馬 with equal dimensions form a cube.

volume. In his commentary, Liu Hui sets out to establish the correctness of this very algorithm.[46] The beginning of his proof is the most important part for us here.

To present his proof, as we have seen in the Introduction, Liu Hui first suggests modifying the values in the problem by considering a *yangma* with dimensions (width, length, height) equal to 1 *chi*. He then remarks that three such *yangma* form a cube, as is shown in Fig. 5.

It follows immediately that the volume of the *yangma* is one-third of the volume of the cube with the same dimensions, which is obtained by multiplying its length, width, and height by one another.

However, Liu Hui notes—and this is the key point for our purpose—that this reasoning does not extend to the general case, when the dimensions are not all equal. The reason for this is that when the dimensions are different, the three pyramids into which the parallelepiped with the same dimensions can be decomposed are not identical, as was the case for the cube. Thus from this decomposition, one cannot conclude that the volumes of the three pyramids are equal before the procedure under study has been proved correct. As a result, Liu Hui discards the first reasoning and starts a second one, general and much more complex.

So far, one may think that the change of the three numerical values of Problem 5.15 to 1 *chi* was intended to focus the reader's attention on a specific preliminary case. Surprisingly, in the general reasoning, instead of making use of the values of Problem 5.15, or of no values at all, Liu Hui again introduces three other values for the length, the width, and the height, this time making each equal to 2 *chi*. Why does he change the values of the problem at all? Why does he change the values he has just introduced to new values that are also equal to each other? We shall sketch the first and main part of Liu Hui's general proof—the only one important for the points we want to make—to find answers to these questions.

For the second reasoning, Liu Hui considers simultaneously a *yangma* and a specific tetrahedron named *bienao* 鼈臑, which together form a half-parallelepiped called *qiandu* 塹堵 (see Fig. 6). Both solids, the *qiandu* and the *bienao*, are the topics of problems of *The Nine Chapters* itself (5.14 and 5.16, respectively). On the basis of the *qiandu* thus formed, Liu Hui sets himself a new goal: establishing that the *yangma* occupies twice as much volume in the *qiandu* as the *bienao*. From this property, the correctness of the procedure for computing the volume of the pyramid can be derived immediately.

However, we do not need to examine this part of the reasoning here, since to make our points, it suffices to evoke the beginning of Liu Hui's reasoning, whereby he establishes the proposition that is his new target.

[46] This commentary was the topic of quite a few publications, among which are Li Yan [1958, 53–54] (I could not consult the first edition of this book, but the second sketches the essential points of the proof), Wagner [1979], Guo Shuchun 郭書春 [1984a, 51–53], and Li Jimin 李繼閔 [1990, 295–303]. See also Chemla [1992b], Chemla and Guo Shuchun [2004, 396–398, 428–433, 820–824], in which a French translation is provided. Note that Wagner [1979] provides a full translation of the commentary into English. I refer the reader to these publications for greater detail, concentrating here only on the features that are important for my argument.

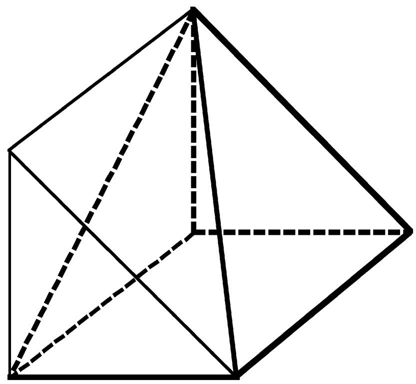

Fig. 6. A *yangma* and a *bienao* form a *qiandu* (half-parallelepiped).

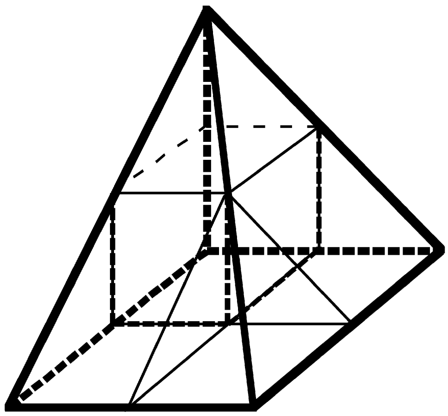

Fig. 7. Halving the dimensions of the *yangma* pyramid decomposes the solid into unit blocks.

His first step consists in forming the *bienao* and the *yangma* with dimensions all equal to 2 *chi* from blocks of the following types: parallelepiped, *qiandu*, *yangma*, *bienao*, all with dimensions of 1 *chi*.[47] Let us describe the composition of the *yangma* and the *bienao* successively.

The *yangma* is decomposed into several kinds of pieces, the dimensions of which are all half of the dimensions of the original body (see Fig. 7). On top, and in front on the right-hand side, we see two small *yangma*, similar to the one we started with. In the back, on the right-hand side, and in front on the left, we see two small *qiandu*, whereas in the back, below and on the left-hand side, we have a small parallelepiped. This is how, for space geometry, Liu Hui uses blocks to compose solids. The reasoning that follows is typical of Liu Hui's reasonings of that kind.

As for the *bienao*, the original body is composed of two kinds of blocks, the dimensions of which are all half of the original dimensions (see Fig. 8). On top in the rear, and in front on the right-hand side, we see two small *bienao* similar to the one we started with. In front, on the left-hand side, on top of each other, we see two small *qiandu*.

If we bring together the solids thus composed, we obtain a global decomposition of the original *qiandu*, as shown in Fig. 9. I have drawn it with the blocks set apart to make easier the enumeration of the components (see Fig. 10). However, it must be kept in mind that Liu Hui's commentary refers throughout to the *qiandu* as a whole.

Liu Hui considers separately two zones in the *qiandu*.

The first zone is the space in it that is occupied by pieces similar to the original *bienao* and *yangma*, but the dimensions of which are all half of the original ones. We know that there are two such pieces of each kind. Within

47 The *bienao* is formed with vermillion blocks, whereas black blocks are used for the *yangma*. But I will not mention the colors any further here, because they do not matter for the point I want to make. The reader interested is referred to Chemla [1994]. More generally, the reader is referred to the published translations of Liu Hui's commentary on the *yangma* to get a more precise idea of the original text (see footnote 46).

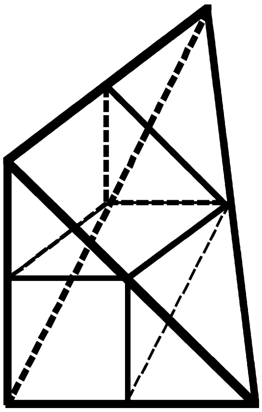

Fig. 8. Halving the dimensions of the *bienao* tetrahedron decomposes the solid into unit blocks.

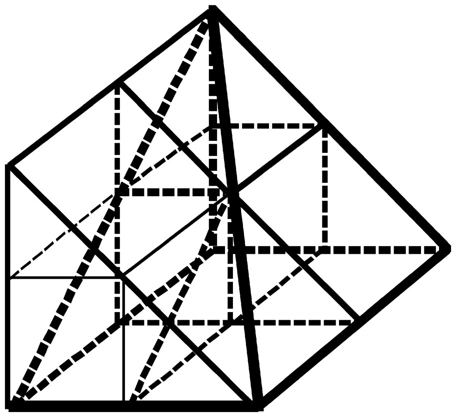

Fig. 9. The half-parallelepiped decomposed.

the *qiandu*, they form, two by two, smaller *qiandu*, one situated in the lower and front part, on the right of Fig. 9 (or Fig. 10, indifferently) and the other situated on the upper and rear part of Fig. 9 (or Fig. 10), on the left.

The second zone, in the space within the *qiandu*, is composed of smaller *qiandu* and a small parallelepiped. All these pieces may be oriented in different ways, but their three dimensions are uniformly half the length, half the width, and half the height of the original *qiandu*. Liu Hui has already established that half a parallelepiped has a volume that is half the volume of the corresponding parallelepiped. The volume of these pieces is hence easy to compute. However, this is not the relevant feature in the situation—and Liu Hui does not mention it. The only useful information is that the volumes of all these half parallelepipeds are the same.

But there are two other features that are essential.

The first key feature derives from evaluating the relative occupation, in the second zone, of blocks coming from the *yangma* and blocks coming from the *bienao*. It turns out that in the second zone, there is twice as much space occupied by pieces coming from the *yangma* as there is space occupied by pieces coming from the *bienao*. The property sought for (i.e., the *yangma* occupies twice as much volume in the *qiandu* than the *bienao*) is hence established in the second zone.

As for the first zone, it replicates, on a different scale, the situation we started with.

The second key feature in the situation concerns the evaluation of the respective proportion of the space in which the situation is known and the space where it still needs to be clarified.

To determine this proportion, by making two cuts in the *qiandu*, Liu Hui removes the upper half-parallelepiped in the front and brings it closer to the similar half-parallelepiped in the lower rear part of the figure. In addition, he removes the rear upper half-parallelepiped, which is composed of a smaller *bienao* and a smaller *yangma*, and moves it near the similar one in the lower front part of the figure (see Fig. 11).

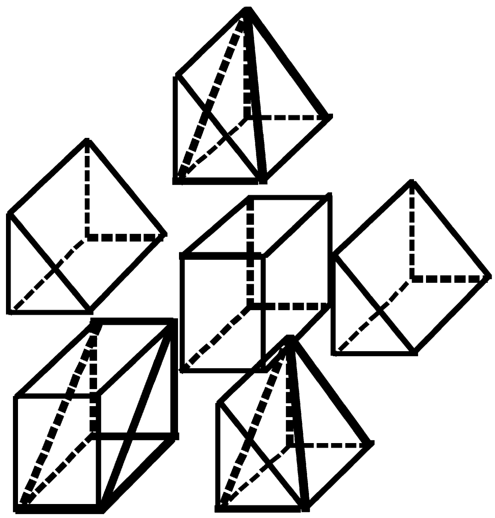

Fig. 10. The half-parallelepiped decomposed, with the pieces set apart.

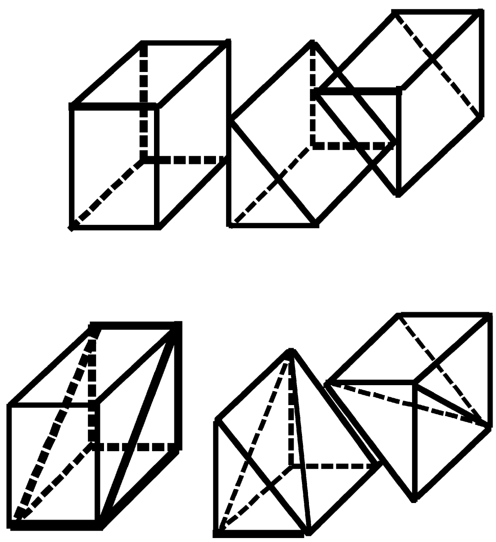

Fig. 11. The original *qiandu* rearranged into four smaller *qiandu*, which in fact compose a parallelepiped.

This rearrangement, which moves only two pieces—two smaller *qiandu*—transforms the original *qiandu* into a parallelepiped, composed of four smaller parallelepipeds, the dimensions of which are all half of the original ones. Seen from this perspective, the second zone considered above—the one in which the property sought for is established—clearly appears to occupy 3/4 of the original body, whereas the first zone occupies only 1/4 of it. These relative proportions ensure that when one repeats the same reasoning, 3/4 of the remaining 1/4 of the original *qiandu* can be shown to hold the property, and so on.

4.2. The proof is general, but it makes use of specific values

To make the points I have in mind about the meaning of the numerical values appearing in the statements of problems, we need not discuss the final part of Liu Hui's general proof, in which he establishes the proportion of 2 to 1 that he chose as his new goal to prove the correctness of the algorithm computing the volume of the *yangma*. We have seen enough of this proof to note how different it is from the proof for the special case. Yet both proofs are constructed using particular values, for which all the dimensions of the solids involved are equal. If Liu Hui was only

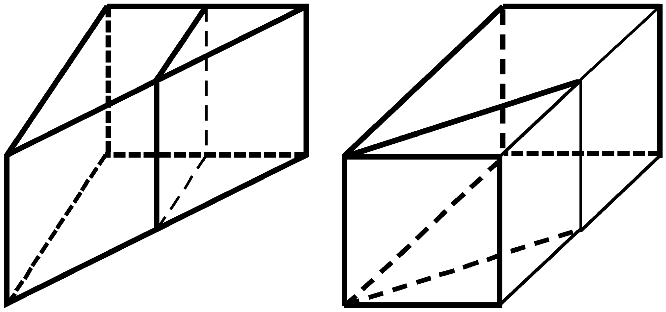

Fig. 12. The *qiandu* and the geometrical transformation underlying the proof of the correctness of the procedure computing its volume.

aiming at making a proof valid for the special case—precisely the one he introduces by his change of values for the dimensions of the *yangma*—he would use a simpler argument: this argument is simply the first argument he makes and then discards as not general.

Two conclusions can be derived from these remarks. To begin with, when Liu Hui first changes the values of the dimensions of the *yangma*, his goal is not to introduce a simpler case for which a straightforward argument can be made to establish the correctness of the algorithm. Second, the second proof, conducted and presented within the framework of the simplest situation possible—that with dimensions all equal—is meant to be general. We are hence led to the same conclusion as we have drawn at the end of the previous section: with respect to the proof, the commentator discusses the general in terms of the particular, and even the most particular possible. The same conclusion holds true for the various problems and procedures, whether they are those from *The Nine Chapters* itself or those used within the context of Liu Hui's proof.[48] In the case with gold, persons, and horses, discussed in Section 3, as well as in the case of the *yangma*, the extension of the validity of the reasoning is determined on the basis of the operations put into play. We meet again with a parallel between problems and visual tools, based on the fact that they are used in similar ways. We shall come back to this remark shortly.

In Section 2, we have shown how the proof of the correctness of the procedure for the trapezoidal prism brought to light that the geometrical transformation needed could be subsumed under the general formal operation of "using the excess to fill up the void." In my view, the way of conducting the general proof for the *yangma* also reveals how it brings into play the same general operation.[49] To explain this point, we need to sketch how, after Problem 5.14 and the procedure for it, Liu Hui comments on the volume of the *qiandu*. His commentary on the algorithm computing the volume of this solid offers two ways of accounting for the correctness of the procedure. The first one relies on the fact that two identical *qiandu* make a parallelepiped. However, there is a second argument, which is not easy to understand. I have suggested that it refers to the *qiandu* as a particular case of the trapezoidal prism Chemla [1992b]. As a consequence, the procedure given for the *qiandu* appears simply to be an application of that for the prism and the proof of its correctness derives from the general proof for the prism, applied to the specific case of the *qiandu* (see Fig. 12).

In turn, the proof for the correctness of the procedures for the *yangma* and the *bienao*, which depends on that for the *qiandu*, fits within this framework, except that the piece to be moved needs to be cut in the middle and the pieces obtained have to be exchanged before they are moved. In this way, again, the operation of "using the excess to fill up the void" appears to be efficient and in fact to underlie the transformations needed to conclude the proof.

From the previous remarks, we can conclude that in the case of the *yangma*, as in the other cases examined above, through proving, the commentator also here seems to aim at identifying general and fundamental transformations underlying a number of different algorithms. Yet, in contrast with the multiplication of fractions examined in Section 3, for which the change of the situation of the problem was linked to the goal of exhibiting the general formal operations underlying the procedure, for the *yangma* the change of the values is not linked to this question. If we are to understand

48 Indeed, as in Liu Hui's commentary on the area of the circular segment, it is clear that in order to unfold, the proof needs the original bodies to be cut into other bodies. These bodies are associated with procedures that compute their volumes, and these procedures are employed in the proof.

49 I make this point in greater detail in Chemla [1992b].

the use of the particular values provided by problems, we must therefore answer the following question: Why does Liu Hui twice change the values of the problem, when clearly any particular values could do?

The answer is clear when we observe the two proofs above: the values of the dimensions used by Liu Hui are determined by the visual tools he used, that is, the blocks, which were concrete objects, all dimensions of which were equal to 1 *chi*. In fact, the first reasoning brings together three identical blocks. In relation to the physical operation carried out, the values needed are all equal to 1 *chi*. The second reasoning illustrates the composition of a *yangma* and *a bienao* with blocks. The same blocks being used, the values needed to refer to the composition of the solids are 2 *chi*. As for the iteration of the decomposition of the *yangma* and the *bienao* in the second step of this proof, its analysis is based on the same configuration as in the first step. Blocks with the simplest dimensions have the property of being polyvalent for all these uses. Using the same blocks, Liu Hui can develop a proof that is particular (the first one) or a proof that is general (the second one).

It is thus the physical features of the objects with which the proof is conducted that dictate the change of values. This conclusion implies that the commentaries were written down by reference to objects used in the course of proving. Note that there is no hint that blocks were used at the time when *The Nine Chapters* itself was compiled.[50] One may assume that these blocks—or some of their uses—were introduced for the sake of exegesis. This hypothesis would explain why the commentator had to change the values to refer to the concrete use of objects in relation to his reasonings, whereas the problems in *The Nine Chapters* would not mention values relating to material objects. In any event, the conclusion reached accounts for the fact that, except for the introduction of new values for a problem employed as a counterexample, the only systematic cases in which the commentators change the numerical values of problems occur in the context of geometry and in relation to the use of material visual tools. The new values are all determined by the objects used as visual tools to compose the body under consideration. The generality of the proofs developed is an issue that is completely dissociated from the fact that the texts could be written as referring to specific material objects. The next point will highlight this conclusion from another angle.

4.3. Visual devices and problems as tools to express the "meaning yi" of operations

Although the particular and the general proofs are different on various accounts, they both use blocks to express the "meaning *yi*" of operations in the same way. In the former proof, gathering the three blocks yields a solid, the volume of which expresses the "meaning" of the multiplication together of length, width and height. The coefficient 3 is interpreted as related to the composition of the cube with three *yangma*. In the latter proof, like in the commentary following Problem 1.36 and bearing on the area of the circular segment, the commentator introduces a procedure that computes iteratively the extension of the space, within the *qiandu*, in which the volumes of the pieces coming, respectively, from the *yangma* and the *bienao* are in the proportion of 2 to 1. In order for its various steps to receive a "meaning," the interpretation of the procedure requires that the volume of the *yangma* be divided. Interestingly, this question of interpretation brings us back to the parallel between problems and visual tools alluded to above.

In the case analyzed in Section 3, a problem was changed into another problem, the situation of which was richer in possibilities of interpretation, since this feature was required for making explicit the "meaning" of the specific operations required by a proof. The same phenomenon recurs here in relation to the *yangma*: the change of values

50 Several points should be stressed about the solids used in the earliest surviving documents and the hints about the early use of blocks. First, in contrast to *The Nine Chapters*, the *Book of Mathematical Procedures* does not mention the *yangma* or the *bienao*, that is, pieces among the blocks essential to establish the correctness of the procedures for other solids. However, the book treats other solids also found in *The Nine Chapters* in ways that require knowing the volumes of the *yangma* and the *bienao*. Several algorithms for volumes in both *The Nine Chapters* and the *Book of Mathematical Procedures* point to a geometrical interpretation, which provides reasons for the correctness of the procedures. This is how the commentators read them in the case of *The Nine Chapters*: they interpret the procedures step by step with the help of blocks [Chemla, 1990]. In such commentaries, Liu Hui also regularly changes the values of the problems and uses blocks in the first way described above, that is, in a way that is not geometrically general. Does this indicate that the commentators knew that objects of the type of blocks were used at the time when our earliest extant sources were composed, but used only in a certain way? Or did they introduce blocks as a tool for exegesis? It is impossible to give a certain answer to this question. However, the generic terms for "figure" or "block" do not occur in the earliest sources. The first known occurrences of these terms are found in the commentaries. Moreover, we may assume that if blocks did not exist, geometrical practice may have been similar to what some commentaries betray within the context of some problems dealing with geometrical topics: instead of referring to specific visual aids, the commentators interpret the operations directly in terms of the physical features of the situation of the problem. See my introduction to Chapter 9 in Chemla and Guo Shuchun [2004, 673–684].

in the problem of *The Nine Chapters* is correlated to the introduction of an inner decomposition of the solid, which creates further possibilities for the interpretation of the steps needed by the proof. Not only does this conclusion offer a similar explanation for the change of a situation and for a change of numerical values. It also leads to an interesting observation. Seen from this angle, the use of a problem and the use of a visual tool are similar[51]: On the one hand, they "illustrate" a situation. On the other hand, they are put into play to express the "meaning" of operations, and when they are not rich enough to support the needs of interpretation required by a proof, they are replaced. In this context, it is worth recalling how the commentator introduces blocks when he first resorts to them: " 'Speech cannot exhaust the "**meaning** *yi*" ' (*yan bu jin yi*),[52] hence to dissect/analyze (*jie*) this (volume), one must use blocks; this is the only way to get to **understanding** (the procedure). 言不盡意, 解此要當以棊, 乃得明耳。" We find here again, in relation to visual devices, the combination of terms (meaning, understanding) that in Section 3 was used in relation to problems.

The previous discussion highlighted why in the context of the commentary on the *yangma* the values of the *problem* were changed into other values, which refer to the dimensions of a *block*: In fact, the continuity between the two kinds of item—problem and visual tool—is thereby manifested by, and inscribed in, the text.

All that has been established for blocks in fact also accounts for the specificities of what is known regarding figures. This is why, even though all the evidence examined here bears on blocks, we have formulated our conclusions for visual devices in general. The perspective developed further suggests an interpretation of Liu Hui's own description of his activity, when, in his preface to the book, he writes: "The internal constitutions (理 *li*) are analyzed with statements (辭 *ci*) and the bodies are dissected with figures (圖 *tu*).[53] 所析理以辭, 解體用圖。" [Chemla and Guo Shuchun, 2004, 126–127].

This concludes what I wanted to establish about Liu Hui's practice with problems. Until this point in the paper, I have mainly concentrated on the commentaries, since they provide evidence to support conclusions about the practice with problems. To which extent can these conclusions on the practice with problems obtained thanks to the evidence found in the commentary be transferred to the mathematical activity at the time of the composition of *The Nine Chapters* itself, or even the *Book of Mathematical Procedures*? This question will be examined in Section 5. With the example of visual tools we have already seen reasons to believe that not all mathematical practices were the same. What about problems? This point is all the more important since in the earliest extant writings problems play a prominent part.

5. Back to *The Nine Chapters*: Connecting the evidence from the commentaries and the Canon

How can one determine whether the editors of the Canon also used problems in a way similar to that of Liu Hui described above? Let us repeat that, unless new sources are found, we will not be able to answer this question with full certainty. The method I will suggest here is to gather hints in *The Nine Chapters* indicating continuities with respect to the practice evidenced by Liu Hui's more prolific writings. I shall allude to some of these hints below. However, another paper would be needed to deal with the question more systematically.

One essential point should be first stressed. Much of what I have said bears on the use of problems within the context of proof. If we believe, as is often stated, that in sheer contrast to the commentaries *The Nine Chapters* contains nothing relating to proof and betrays no interest in this dimension of mathematical activity, this would deny from the outset the possibility of a real continuity. However, I have argued several times that, even though *The Nine Chapters* includes no fully developed proof, various facts indicate that the authors had an interest in understanding why their procedures were correct. We have already indicated some of them. For example, we have noticed that the qualification of division as "dividing in return (*baochu*)" adheres to the sphere of justifying procedures. It is hence

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51 Even though, in his unpublished dissertation, Wang Ling did not analyze in detail how problems and figures were used in the course of proving, it is interesting that he used the same term of "model" to refer to both problems (he speaks of "'model' problems" or "variant models") and figures. Moreover, his conclusion regularly stresses that in the mathematics of ancient China, the particular was used to deal with the general [Wang Ling, 1956, 211, 282, 287, 295]. However, he did not provide evidence to support his claims, which is my main purpose in this paper.

52 The commentator quotes here the "Great commentary" (*Xici dazhuan*) to the *Book of changes* (first chapter, paragraph 12). Compare Chemla and Guo Shuchun [2004, 374–375].

53 Here, probably, the generic term of "figure" stands for all the visual tools used in the commentary. On the terms occurring in the statement, see my glossary.

quite striking that this expression occurs several times in *The Nine Chapters* itself. Why should one prescribe to "divide in return" instead of "divide," unless to indicate the reason for carrying out the operation? In addition, we have mentioned that Liu Hui interpreted some algorithms for computing the volume of solids as indicating a proof of their correctness through the description of the procedure (see footnote 50).

A further hint in support of the claim that the authors of *The Nine Chapters* had an interest in the correctness of algorithms is provided by how Liu Hui appears to interpret the fact that they described procedures in the context of problems. Not only does this piece of evidence indicate that he reads a concern for proof in the Canon, but it also reveals that he links the problems in the Canon to this concern. Let us therefore examine this evidence in greater detail.

In fact, there are cases where procedures in the Canon are described **outside of the context of problems**. Such is the case, for instance, for the rule of three (*jinyoushu* 今有術) [Chemla and Guo Shuchun, 2004, 222–225]. This observation indicates that a problem is *not* an indispensable component in presenting a procedure. It is interesting, incidentally, that, to establish the correctness of the rule of three, Liu Hui's commentary introduces a problem allowing the interpretation of the "meaning" (*yi*) of the operations. This fact confirms from yet another angle the part played by problems in proofs in relation to making explicit the "meaning" of the operations of an algorithm. What is most interesting in this case, however, lies elsewhere. The commentator states about this procedure for the rule of three: "This is a universal procedure (此都術也 *ci doushu ye*)." There is only one other passage in which Liu Hui repeats this statement. There it applies to the procedure for solving systems of simultaneous linear equations (方程術 *fangchengshu*), mentioned in Section 2 above [Chemla and Guo Shuchun, 2004, 616–617]. However, in contrast with the rule of three, in this case the "universal procedure" is described by *The Nine Chapters* within the context of a problem about different types of millet. It reads as follows:

> (**8.1**) Suppose that 3 *bing* of high-quality millet, 2 *bing* of medium-quality millet and 1 *bing* of low-quality millet produce (*shi*) 39 *dou*; 2 *bing* of high-quality millet, 3 *bing* of medium-quality millet and 1 *bing* of low-quality millet produce 34 *dou*; 1 *bing* of high-quality millet, 2 *bing* of medium-quality millet and 3 *bing* of low-quality millet produce 26 *dou*. One asks how much is produced respectively by one *bing* of high-, medium- and low-quality millet.[54] 今有上禾三秉,中禾二秉,下禾一秉,實三十九斗;上禾二秉,中禾三秉,下禾一秉,實三十四斗;上禾一秉,中禾二秉,下禾三秉,實二十六斗。問上、中、下禾實一秉各幾何。[Chemla and Guo Shuchun, 2004, 616–617]

The piece of evidence in which I am interested is found immediately after the sentence in which Liu Hui qualifies the procedure as being "universal." The statement he adds to this can be interpreted as accounting for why, here, in contrast to the case of the rule of three, the Canon uses the context of problems to present the procedure. The commentator writes:

> It would be difficult to **understand** (the procedure) with abstract expressions (*kongyan*), this is why one deliberately linked it to (a problem of) millet to eliminate the obstacle. 以空言難曉,故特繫之禾以決之。[55]

This statement reveals that, in Liu Hui's perspective, the purpose of the Canon for presenting a procedure in the context of a problem was related to the aim of having the procedure be understood. We have already shown the relationship between "understanding" a procedure and establishing its correctness. In the text that follows, Liu Hui uses the context of the problem on millets to interpret the operations of the procedure and thereby bring to light that

[54] *Bing* 秉 designates a unit of capacity from a system of units different from that of the *dou*. I am grateful to Michel Teboul, who suggested another interpretation for this problem: the statement may be understood as referring to distinct units of capacity all named *bing* and the value of which would depend on the grain measured. I have shown that in *The Nine Chapters*, the Canon gives evidence of a similar phenomenon with respect to the unit of capacity *hu* 斛, which had different values for different grains [Chemla and Guo Shuchun, 2004, introduction to Chapter 2, 201–205]. The key term *shi* in Problem 8.1 could thus be interpreted either as "fill up" or as "capacity" (the basic sentence would then read: "Suppose that 3 *bing* of high-quality grain, 2 *bing* of medium-quality grain, and 1 *bing* of low-quality grain (correspond to) a capacity (*shi*) of 39 *dou*"). The meaning of *shi* as "capacity" is attested in a passage from the *Records on the scrutiny of the crafts* (*Kaogongji* 考工記, dated to the third century B.C.E.), which deals with standard vessels and is quoted in Liu Hui's commentary on Problem 5.25 [Chemla and Guo Shuchun, 2004, 450–453]. However, the actual values in the problem still make me prefer the first interpretation, which I have therefore inserted here.

[55] On the interpretation of *kongyan*, see Chemla [1997a] and the entry for *kong* "abstract," in my glossary [Chemla and Guo Shuchun, 2004, 947]. Chemla [2000] develops a reading of Chapter 8 of *The Nine Chapters* along these lines, that is, by assuming that the millets were intended to enable an interpretation of the operations in the procedures.

the pattern of "equalizing" and "homogenizing" underlies it, thus proving its correctness (see Section 2). The piece of evidence that the above statement constitutes may hence indicate that Liu Hui reads the Canon as providing problems to be put into play to prove algorithms.

Several additional features of the problems used in the same chapter are quite interesting to support our argument that the practice of problems in *The Nine Chapters* presents continuities with Liu Hui's commentary [Chemla, 2000].

First, in fact, the statement of Problem 8.1, regardless of whether one interprets it in terms of production of millet or of capacities (cf. footnote 54), can be interpreted in a still completely different way:

> (**8.1, alternative interpretation**) Suppose that 3 *bing* of high-position millet, 2 *bing* of medium-position millet and 1 *bing* of low-position millet correspond to the dividend (*shi*) 39 *dou*; 2 *bing* of high-position millet, 3 *bing* of medium-position millet and 1 *bing* of low-position millet correspond to the dividend 34 *dou*; 1 *bing* of high-position millet, 2 *bing* of medium-position millet and 3 *bing* of low-position millet correspond to the dividend 26 *dou*. One asks how much is the dividend corresponding respectively to one *bing* of high-, medium- and low-position millet.

In addition to interpreting "high," "middle," and "low" as positions on the calculating surface, this second reading interprets the term 實 *shi* ("produce"/capacity) with its technical meaning in mathematics, "dividend." From the first century C.E. till the fourteenth century at least, in ancient China an algebraic equation was conceptualized as an opposition between divisors (the various coefficients of the unknown) and a dividend (the constant term). In fact, it can be shown that Liu Hui *also* reads the statement of Problem 8.1 in this alternative way and understands a linear equation as opposing a dividend (its constant term) to divisors.

This first problem of Chapter 8 is followed by five others similar to it, all relating to grain. The full procedure for solving systems of linear equations, with positive and negative numbers, is then unfolded in relation to them.

If Problem 8.6 were interpreted along the same lines as the previous ones, it would correspond to a system of equations with two negative constant terms. However, all ancient sources give the two "productions/dividends" as positive, a fact that the commentators do not stress as erroneous. In other terms, it seems that the interpretation of the "dividends" in terms of "production" or "capacities" imposes that the *shi* remains positive. If this is the case, this constraint, which derives from the interpretation of the operations in the terms of the problem, poses limits to the full presentation of the mathematical topic. What follows in *The Nine Chapters* supports this hypothesis: Problems 8.7 and 8.8 turn to a new situation—buying animals—to discuss new types of systems in which dividends—in this context no longer "productions" or "capacities"—are either positive, negative or zero.

This seems to indicate that in *The Nine Chapters* the situation with millet producing grain (or contained in units of capacities) was used to deal with systems of equations as long as the interpretation allowed by the situation did not conflict with the mathematical requirement. As soon as a divergence arose between the interpretation in the terms of the problems and the mathematical meaning, the situation for discussing the topic was changed to allow further discussion. If our interpretation is correct, this implies that, in agreement with Liu Hui's explanation, the use of the situation of a problem for interpreting the operations of an algorithm, in relation to understanding and proving it, dates to the time of *The Nine Chapters*.

Such a conclusion suggests that there may have existed a mathematical culture of situations selected for their usefulness in presenting procedures. Another element appears to confirm this hypothesis: many mathematical questions are discussed within the framework of the same kind of situation throughout the centuries. One way of accounting for this would be that these situations proved particularly suitable in relation to interpreting the operations of the procedures. Inquiring further into this question exceeds the framework of this paper. Let me simply claim for now that this is quite plausible.

6. Conclusion

We have seen that not only does the commentator Liu Hui attest to a practice of problems, described in Sections 1 to 4, that is peculiar and differs from the one most common today, but he also seems to assume that his way of using problems was in continuity with former practices, especially those contemporary with the making of the Canon. In fact, we have found hints in *The Nine Chapters* that appear to support his belief. In this tradition, whether problems were practical in form or abstract, they were read as general statements, the extension of which was determined on the basis of the procedure relating to them. This fleshes out what the opening sections of *The Gnomon of the Zhou* intended when depicting intellectual activity in mathematics or astronomy as aiming at widening "classes (*lei*)" of problems. Instead of interpreting these statements on the basis of our own experience, it seems to me more appropriate to rely on the evidence we have about mathematical activity in ancient China to interpret this early theoretical description of what mathematics was about. Furthermore, the results obtained in this paper give us elements for establishing a method to interpret our earliest Chinese sources.

As we have seen, the procedure given for a problem can either solve it or be used for establishing the correctness of another algorithm. This is why problems can be either questions to be solved or statements describing a situation in order to interpret the meaning of operations. Clearly, ignoring such facts would be quite detrimental when reading mathematical sources from ancient China. This is one of the main errors responsible for the mistaken idea that these texts can be adequately interpreted as merely practical. However, the benefit of promoting a new method of reading is not limited to this aspect. Instead of assuming that mathematical practice has been uniform in space and time, such a way of approaching texts, attempting to establish how they should be read before one sets out to read them, contributes to restoring the diversity of mathematical practice. I hope the above arguments will inspire further research that by gathering specific evidence will enable us to restore various practices of mathematical problems. Once we have assembled several similar case studies, we shall be in a position to outline a research program that may create new conditions for interpreting our sources.

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