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ON A PROBLEM OF GRAPH THEORY
by
P.ERDŐS,A. RÉNYII and V. T. SÓS2
§0. Introduction
Let G„ be a non-directed graph havingnvertices, without parallel edges and
slings. Let the vertices ofGnbe denoted by F1,...,Pn.Let v(Pj) denote the valency
of the point Piand put
(0. 1)

V(G,)=maxv(Pj).
1ninn
LetE(G.)denote the number of edges ofGn.LetHd(n, k)denote the set of all
graphsGnfor whichV(Gn)= kand the diameterD(Gn)of which is--d,
(k=1, 2,...,n-1; d=2, 3,..., n-1).
In the present paper we shall investigate the quantity
(0.2)

F, (n, k)=minE(Gn).
G„ E Ha(n, k)
Thus we want to determine the minimal number N such that there exists a graph
havingnvertices,Nedges and diameter--dand the maximum of the valencies
of the vertices of the graph is equal tok.
To help the understanding of the problem let us consider
the following interpretation. Let be given in a country n airports;
suppose we want to plan a network of direct flights between these
airports so that the maximal number of airports to which a
given airport can be connected by a direct flight should be equal
tok (i.e.the maximum of the capacities of the airports is pres-cribed), further it should be possible to fly from every airport
to any other by changing the plane at mostd-1times; what is
the minimal number of flights by which such a plan can be
realized? For instance, ifn = 7, k = 3, d= 2we have F2(7, 3) = 9
and the extremal graph is shown by Fig. 1.
The problem of determiningFd(n, k)has been proposed and discussed recently
by two of the authors (see[1]).In § 1 we give a short summary of the results of the
paper[1],while in §2and3we give some new results which go beyond those of
[1].Incidentally we solve a long-standing problem about the maximal number
of edges of a graph not containing a cycle of length 4.
In § 4 we mention some unsolved problems.
Let us mention that our problem can be formulated also in terms of 0 -1
matrices as follows: LetM=(ail)be a symmetricaln by nzero-one matrix such
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Fig. I
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thatsii=l,max

El].=k+1and all elements of Mdare ~- 1. We want to
1~i=nj=1
determine
Clearly
(0.3)
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Md(n, k)= min
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Md(n, k)=2Fd(n, k)+n.
This formulation shows the connection of our problem with non-linear programming.
We give for the cased=2a third formulation of our problem which displays
its connection with the theory of block designs.
Let be given a sequenceA,, A2,..., Anof subsets of the elements 1, 2,..., n
such that ifjEALtheniE Aj.Let us suppose that denoting byJAIthe cardinal number
of the setA,we havemaxAjI=k.Let us suppose that for anyi(I -_i-_n)and
1-~j~n
any jisuch thatjJALthere is a setA,,which contains bothiandj(this is equivalent
by our supposition of symmetry to the statement that the setsAiandAjare not
disjoint). The problem is to determine
(0.4)

min~Ai~= 2F2(n, k).
=1
§1. Some Basic Inequalities, and some Asymptotic Results
It is easy to see that if there exists a graph GnwithV(G.)=kand diameter
d, then
(1. 1)

n -I+k
k-2
(1. 1) can be proved as follows: ifV(GJ=kthe number of points which can be
reached from a given point, say, P1by an edge is--k;the number of points which
can be reached from Plby a path of length2 isk(k-1)and finally the number
of points which can be reached by a path of lengthd is-k(k-1)d-1.Thus if the
graph has diameter--dwe have
(k-1)d-1
(n-1)-k(1+(k-1)+(k-1)2+...+(k-1)d-1).
This proves (1. 1). If bothnandkare odd, thenGnmust contain at least one
point of valency--k-1(because the number of points of odd valency cannot be
odd); thus in this case we get
d
(1.2)

n--1+(k-1)
(k_kl)2_
I.
Note that for the graph shown by Fig. 1, equality stands in (1. 2). For the graph
shown on Fig.2(the so-called Petersen-graph) equality stands in (1. 1) withn =10,
k=3, d=2.

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As regards F,,(n, k) we obtain easily the lower bound

(1.3)

Fd (n, k) --

n(n-1)(k-2)

2((k- 1)`i - 1)

(1. 3) can be proved as follows : every edge is itself a path of length 1 ; it can be contained in at most 2(k-1) paths of length 2, but in this way each path of length 2 is counted twice, thus the num-

ber of paths of length 2 cannot exceed E(G„) (k - 1).

In general each edge can be contained in at most
3(k- 1) 2 paths of length 3, but in this way each
path of length 3 is counted three times, thus the
number of paths of length 3 cannot exceed
E(G„) (k-1) 2, etc. As in case G„ has diameter

d the number of paths of length d has to be at least (2 , we obtain

(1.4)

E(G„)(1+(k-1)+ .. . +(k- 1)d-1)

lJ

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Fig. 2

which implies (1. 3). Note that one has equality in (1 . 4) for the Petersen graph shown

on Fig. 2., further for n=5, k=2, d=2 because a cycle of length 5 has 5 vertices,

each of which has valency 2, it has diameter 2 and the number of its edges is 5 = 2. 2.

It is clear from the above proof that one can have equality in (1. 4) only for a

regular graph of order k, i. e. if E(Gn) = 2 and if any two points are joined by

one and only one path of length -- d.

The first condition implies that if equality stands in (1 . 4) then there is equality
in (1. 1) too. For the case d=2 this means that a necessary condition of equality

in (1. 4) is n =k2 + 1. It has been shown by A. J. HOFFMAN and R. R. SINGLETON [41
that a regular graph of order k, having k 2 + 1 points and diameter 2 exists only

for k = 2, 3, 7 and perhaps for k = 57. Thus for d = 2 except for these values of
k one has strict inequality in (1. 3). However it has been shown in [1] that there
exists an infinite sequence of pairs (k,, n) such that k, -> -, n,

and

(1. 5)

lim

F2 (n,:, k,) k; - 1

,-- n,(n,- 1)

2 '

This is a consequence of the following

THEOREM 1. If P is any prime power, there exists a graph G„ of order n = P 2 + P + 1

for which V(G„) = P + 1, which has diameter 2 and for which E(G„) _ 2 (012 + n).

The graph G„ has also the property that it does not contain any cycle of length 4.

To make this paper self-contained we reproduce the proof of Theorem 1 given
in [1].

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PROOF OF THEOREM 1.LetGF(P)be the Galois field withPelements. Let
us represent the points of the finite plane geometry.PG(P, 2)by triples(a,b, c)
wherea,b,c are elements ofGF(P),not all three equal to 0, and(~a,),b,~c)with
E GF(P),z 0represents the same point as(a,b,c).The number of different
points ofPG(P, 2) is P2+P-{-1.Astraight line inPG(P,2) is the set of all points
(x, y,z) which satisfy the equationax +by+ cz = 0;we denote this line by[a,b, c].
The point(a, b, c)and the line[a, b, c]are clearly conjugate elements with respect
to the conic x2--y2á-z2=0.As well known there areP+ 1points on each line,
any two different lines have exactly one point in common and through any two
given points there is exactly one straight line. Now we define the mapping T which
maps the pointA = (a,b, c)into the linea = [a,b, c]and conversely. We write
TA =a, Ta=A.This mapping has evidently the properties: if the pointBlies on
the linea = TAthen the pointAlies on the line(3 = TB;if C is the point of inter-section of the linesTAandTBthenTCis identical with the line passing through
the pointsAandB; A = (a,b, c)isonTAif and only ifa2+b2+c2=0,i.e. ifA
lies on the conic x2+y2+z2=0. Now let us define a graph G„(n=P2+P+1)
as follows: the vertices ofGnare the points ofPG(P,2);the verticesA= (a, b, c)
andA'=(a',b', c')are joined inGnby an edge if and only ifA'is lying onTA(and
thusAis lying onTA').Clearly a vertexAinGnhas the valencyPorP+ 1according
to whetherAis on the conicx2+y2+z2= 0or not.VThus
and
2
(n'12-n)
z
P(P2+P+1)--F(GJ
E(GJ-1
(P+1)(P2+P+1),2(n3,12+n).
Finally the diameter ofG,, isequal to 2. As a matter of fact any two points
AandBcan be joined by the pathACBwhere C is the point of intersection of the
linesTAandTB.Besides thisAandBcan be joined by a single edge ifAlies onTB.
But the point C such that the edgesACandBCboth belong toGnis in any case
unique; thusGndoes not contain any cycle of length 4.
Thus our Theorem is proved.
We deduce from Theorem 1 the following corollaries.
COROLLARY 1.Putnk= k2-k+ 1;then
(1.7)fiminf
F2(nk,k)k-1
k--nk(nk-1)

2
"If P is prime, there are P+1 points on the conic and thus
)z1
E(G„)=

+
~2
11112if
2

n~nn.
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PROOF OF COROLLARY 1 . By (1 . 3)

(i. 8)

F2 (n, k) k, 1

n(n-1) - 2

further if k = P + 1, nk = P2 + P + 1, by Theorem 1

(1.9)
F2 (P2 -I p+1, P+1)~

1 (P+1)(P2+P+1);

thus in this case

F2 (nk, k) k

1 1

(1. 10) nk (nk -1)

2 I+- ,

this proves our assertion.

Theorem 1 enables us also to solve - at least asymptotically - a problem

which was raised by one of us 27 years ago (see [2]).Y

Let C„ denote the class of graphs having n vertices and containing no cycle

of order 4. Put

(1. 11)
µ(n) = max E(G„).

G„ E C„

The problem is to determine the value of µ(n). From Theorem 1 we deduce the

following

COROLLARY 2. We have

1 (1. 12)

lim

µ n =
n--

n312

2

PROOF OF COROLLARY 2. It follows clearly from Theorem 1 that if P is a

prime power, then putting n = P 2 + P + 1

(1 .13)

µ (n)

- 2 (n3/2 - n).

It is possible that for these n the graph of Theorem 1 is extremal but we can-

not prove this. Clearly µ(n) is an increasing function of n, and thus it follows that

for any n we have

(1.14)

u(n)

Z

[P2+P+1)312-(P2+P-{-1)]

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where P is the largest prime power such that p2 + P + 1-_n.

n--n 1 one can choose a prime p so that

n

(1.15)
vn - log

n

p

} n -1

Now evidently for

After having written this paper we have been informed by W. G. BROWN that independ-

ently of us he has proved (1.12), in the same way as we did. His paper will be published in the Bul-

letin of the Canadian Mathematical Society .

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which implies forn~ni
n1-lo
2
gn)p2+p +1 ~- n.
Thus we have for anyn-ni
(1.16)

µ(n) -2
n3/2
(1-l0g3n)
and thus
(1.17)

liminf
µ(n)1
n--n32

2
On the other hand it is easy to see (this follows also from the results of1. REIMAN
in[3])that
(1. 18)

lim supµ(n)1
J?--n312

2
As a matter of fact, letGnbe a graph containing no cycle of order4.Let P1,P2,...,P„
be the vertices ofGnand let us denote their valencies by vl, u2,..., vn.Now clearly one
can select from the set Eiof vertices joined by an edge to Pi(2`pairs, and no
pair(Pj, Ph)can be contained in bothEiandE,withlibecause otherwiseP; PjPIP,,
would be a cycle contained in G,,.Thus we must have
(1.14)
Now we have
(1.20)
and thus
(1.21)
n
As clearlyfvj=2E(G„),we have
(1.22)
which implies
(1.23)
Thus
(1.24)
Z
n
li(n)
n312-2
„(v1

í
2
n).
2n
4E2(Gn)-2nE(G„)=n3
n3/2
E(Gn)-	
2

1
which implies (1. 18). Thus Corollary 2 is proved.
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(J

2n/2)

n3.
1n
4n+4'
1

1
+
1
4n4.~n
.

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Let us note that weaker results have been obtained previously by E. KLEIN

(see [2]) and 1. REIMANN [3], who proved lim inf

µ(n)

1 REIMANN's extremal

n-- n

2 j 2,

graph does not contain triangles either ; it is possible that among such graphs it is optimal.
Note that for the pairs (n j, kj) for which according to Corollary I one has

(1.25)

lim

F2 (npkj)k; - I

;_,_ nj (n j -1)

2

one has kj -Vn j. It was shown in [1] that there exists another sequence of pairs
(k,, n j) such that

(1.26)

Jim

Fz(n'

, k'-)kj
= 1

j _,_ nj (nj -1)

a
but for this sequence of pairs one has lim ~~' _ + ~.
nj
It remains an open question what is the value of the function g(c) defined by

(1.27)

g(c) = liminf

F2 (n, k) k

k2,n, n(n- 1)

n-~

for I < c +-; we know only that g(c) is nondecreasing, z - ~g(c) and lim g(c); 1.

§ 2. Some Exact Results for d=2.

In this § we deal with the exact value of F, (n, k) for 2

k n - 1. Evidently,

Fz, (n, n -1) =n -1, because the graph Gn in which one vertex is joined by an edge

with all others, has diameter 2, further V (G.) = n - I and E(G,,) = n -1. It has
been shown in [1] that Fz (n, n - 2) = 2n - 4 (a graph G„ with V (G.) = n - 2 and
E(G„)=2n-4 and having the diameter 2 is shown by Fig. 3; another graph with
the same properties is shown by Fig. 4), further that FZ (n, n - 3) = FZ (n, n - 4) _

= 2n - 5. (The corresponding extremal graphs are shown by Figs . 5 and 6.)

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Fig. 4

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We shall prove now

THEOREM 2. We have for n --13

(2.1)

F2 (n, k) = 2n-4 for

2n-2

3
k n-5.

Fig. 6

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Fig. 5

PROOF OF THEOREM 2. The extremal graph G„ with

V(G„)=k=n-1,

15<1-

n 3 2 1

and E(G„) = 2n - 4 and having diameter 2 is exhibited by Fig. 7.

V(G„)=n-1, E(G,)=2n-4, 5,1~

n3

n - 13.

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Fig. 7

Note that all vertices of G„ exceptP,, P2and P3have the valency2,further
v(P,)=n-1, v(P2)=n-1, v(P3)=21-2and by supposition21-2-n-1.Thus
V(Gn)=n-1.Clearly G„ has diameter2and the number of edges of Gnis
E(G„)
=2(n-/)+21
2
-2+2(n-3)
= 2n-4.
We prove that for anyGnwithn ~ 13,V(G„)=n-l 5

n+21
3	and diameter
2one hasE(Gn)--2n-4.
n
As E(Gn) = 75,v(Pi)we may suppose that G„ contains at least one point
2j=1
of degree3.IfGnwould contain no point of degree--2,then let us choose a point
of degree3;let this point beP,.Let the points connected by an edge withP,be
denoted byP2, P3andP4.As every point can be reached fromP,by a path of
length--2,we must havev (P2)+v (P3)+v (P4)-n-1.
Now if there would be a point among the pointsPs,..., Pnwhich would be
connected with more than one of the pointsP2, P3, P4we would havev(P2)+
+v(P3)+v(P4)Win;as all other points have degree~3it would follow
n
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v(Pi)n+3(n-3) = 4n-9
=1
and thusE(Gn)>2n-5i.e.E(Gn)~2n-4,which was to be proved. Thus we may
suppose that all points Pi(5 --i--n)are connected with one and only one ofP2iP3
andP4;similarly we can suppose that P2iP3and P4are not connected with each
other because this would againimply v(P2) +v(P3) +v(P4)--nand thusE(Gn)1-
2n-4.If there is at least one among the P, with5 --i-nwhich has degree >3,
it again follows that E(G„)--2n-4.If however all have degree3,let us suppose
thatV(P2)=min(v(P2),V(PA v(P4))which impliesV(P2)n3I.Let Psbe
connected with P2.Thenv(NS3 and let the three points connected with P, be
P2,P;,and P,;clearlyi>5andj>i>5.But thenv(P2)+v(Pj)+v(Pj)--n-1
and thus
2(n-1)
6 = v (P) + v(P,)

3
that isn10.
As we supposedn13,this case is settled.
The case when there is a point P, of valency 1 is easily settled, because if this
point isP,,and P, is connected with P2only, thenP2has to be connected with
the remainingn-2points too, and thus would have valencyn-1.Thus the only
case which remains to be settled is whenmin v (Pi) =2.Supposev (P,) =2and
1-inn
letP,be connected withP2and P3. Then all remaining points have to be connected
either with P2or with P3or with both.
Let C, denote the class of points Piwithi4 connected only with P2,and
c, the number of elements of C,; let C2be the class of points Piwithi--4connected
only withP3and c2the number of elements of C2;finally let C3be the class of points
connected with both P2and P3,and c3the number of elements of C3.Clearly
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c, + c2 + c 3 = n - 3. As the valency of P 3 cannot exceed n - I and P 3 is connected

with every point in G„ except itself and the points in C,, we have c l --I-2 3.

Similarly c 2 ~-- 1-2-3.

The number of edges in G„ the existence of which is already established is

clearly c l + c2 -P 2c 3 + 2 = n + c 3 -1. Let us call these the edges of the first kind,

and the remaining edges those of the second kind . As the graph has diameter 2,

every point of C, has to be connected by a path of length - 2 with every point of C 2.

Such a path can not contain an edge of the first kind. Thus the graph G' consisting

of the edges of the second kind has to be connected . Now three cases are possible.

Either G' contains besides the points of C, and C 2 at least one further point from

the class C3; in this case it contains at least c, + c 2 + 1 points and thus there are

at least c 1 +c2 edges of the second kind, and thus the total number of edges is

E(G„) --n + c3 -1 + c l + c2 = 2n - 4. Or P2 and P3 are connected by an edge;

in this case we get again E(G„)-_2n-4. Or P2 and P3 are not connected and G'

consists only of the points of C, and C 2. In this case the connected graph G' is

either a tree or not. If it is not a tree, it contains at least cl + c 2 edges and thus we

obtain again E(G„) --2n -4. If G' is a tree, it must have at least two end-points . We

may suppose that C, contains an endpoint of G' . Let x be the total number of end-

points of G' in C 1. Then the sum of valencies (in G') of the points of C, is at least

x + 2 (c, - x). As G„ has diameter 2 and P 2 and P 3 are not directly connected, any

endpoint of G' in C, has to be connected by a path of length 2 to P 3 , it follows that

for every endpoint P of G' in C, the single edge starting from P ends in C 2. Let

y denote the number of points in C 2 which are connected with an endpoint of G'

in C, . If Q is such a point, clearly Q has to be connected with every other point

of C 2, because otherwise there would not exist a path of length 2 from P to

these points. Now clearly no point of C 2 can be an endpoint of G', because it must

be connected to at least one point in C, and also to Q. Thus the sum of valencies

in G' of the points of C 2 ist at least 2(c2-Y)+Y(c2-1)+x. It follows that the

number of edges of the second kind is at least

2 (x+2(c,-x)+2(c2-Y)+Y(c2-1)+x) = cl+c2+Y(

e22 31
' c l +c2,

because, as we have shown, c 2 --3

.

Thus we have shown that E(G„) 2n -4 and the proof of Theorem 2 is complete.

Note that the restriction n =13 in Theorem 2 is necessary, because for n < 13

there is no value of k between

2n-2
3 and n - 5.

As regards the value of F2 (n, k) for k <

2n
3 - -

2
we can show that for n' 15

-k-6

for

3n-3 k < 2n-2

5 3

(2.2) F2 (n,k -4k-10 for

5n- 3 k 3n-3

5

4n - 2k-13 for

n +I1 _ k ` 5n9 3

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We give in what follows the extremal graphs for these3cases. That these are really
extremal can be proved in a way similar to the proof of Theorem2,therefore we
leave the details to the reader.
The graph has four points of high degree; let us denote them byA, B, C,D
and four groups of points.
There is a group denoted byAB,the points of which are joined toAand toB.The group contains2k-npoints. In the groupBCD(connected withB, CandD)
there aren -k-1points. In the groupAC(whose points are connected withAand C)
there are
n-k-32

points; finally in the groupAD(the points of which are con-
--
nected withAandD)there aren - k - 3 - n k 3 2

points. Further the graph
contains the edgesAB, AC, AD.The pointsAandBhave the degreek.The whole
graph has3n - k - 6edges.
THE EXTREMAL GRAPH FOR
5n93k< 3n5 3
There are5points of high order,A,B, C, D, E.The groupABhas2k-npoints,
The groupBCDhas
n-Ic-1
2	points.
The groupBCEhasn - k-1
_[n-k-21

points.
The groupAChas2k-npoints.
The groupADEhas2n-3k- 4points.
Further the edgesAB, AC, AD, AE, DEbelong to the graph. The pointsA, B
and C have the valencyn-k;the total number of edges is5n-4k-10.
THE EXTREMAL GRAPH FOR
n21k<5n93
There are6points of high order,A, B, C, D, E, F.The groupABcontains2k-npoints.
The groupBCEcontains
n-k-1 2

points.
The groupBDFcontainsn -k - 1 - n-k-1 2

points.
The groupADCcontains
n-k-5
2

points.
The groupAEFcontain;n - k - 5 -	
n-k-5
2	points.
The graph contains further the edgesAB, AC, AD, AE, AF.The graphhas4n- 2k-13
edges.
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For k <

n +I

I we cannot determin; Fz (n, k) c xactly. However, we can get

a fairly good upper bound by constructing graphs of diameter 2 by the following

principles. We divide all but

(

2

r)

of the points of a graph G„ into r groups of

approximately the same size . We connect the points of each pair of groups withh

one of the remaining points, and connect as many of these point3 with each other

ae

z ze 2

P

Fig. 8

as needed. For instan ve if r = 4, n=41+6, we put l points in each of 4 groups, connect

each of the 6 pairs of groups with one of the remaining 6 points, and connect eachh

of these points with that point which is connected with the other two groups . The,

graph obtained is shown by Fig. 8. It follows that

(2.3)

F2 (41+6, 21+1)--121+3.

§ 3. Some Remits for d 3.

We prove first

THEOREM 3. We have for every n, every k -. n -1 and d - 3

3

2

(3.1)

Fd (n, k)

kn

I 1-

(3. 3)

PROOF OF THEOREM 3. Let us put

Clearly we may suppose 6 < 1, because otherwise (3 . 1) is trivially fulfilled. We have,

evidently

4

k d

/ 3

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We may supposen > kd1,because any graphG„with diameter -dis connected
and thus has at leastn-1edges; thusF3(n, k) n-1and ifn--Ica-ithe inequality
(3.1) is trivial. Thus we have to prove(3.1) only forkd-1< n <
64 i
.e.for(64n)IId <
<k<n1Id-i
Let G„ be a graph having n vertices, diameterdand such thatV(G„)=k.
Let us denote by Xl,..., XSthose vertices of G„ the valency of which is<
kd4n
6
let Yi,..., Y„_sbe the remaining vertices ofG,,.We have clearly
(3.4)
Thus if
we have
(3.5)
1
E(G„)=21-1
Thus we have to consider only the case
~~-S
(Xi)-I-~ v(Yj)I>?kd	_l~n•
s=n1-

2
z
E(G„)
.n
kd_1
-
(3.6)

s::-n 1-
b(1
2
b)
We distinguish two cases. Either every X,(1--i--s) is connected with at least
~1 -2]~_rof the verticesY,,or not. In the first case we haved
z
(3.7)

E(G„)--s 1-2	
kd
1-(1 -b)kd_,.
Thus we may suppose that there is an X, -say Xi-which is connected with less
thanI1 -
Z~ kd
1 Y,-s.We shall show that this case is impossible. By supposition
we can reach, starting from Xl, every vertex ofGby a path of length--d.Let us
consider first those paths starting from Xl,the next vertex of which is an Y;.
As Yjcan be chosen in < 1-
Z
~,ways, and all vertices ofG„have valency
.k,the number of such pathes is at most
(3.8)

	
I1-2)kd
i(1-+-(k-1)-+-(k-1)z+...+(k-1)d-i)Z~n.
*We may also suppose that k-64.
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Let us count now the pathes of length d starting from X1,on which the point
next to XI,is anXi.The number of such pathes is clearly at most
(3.9)

kd%(1+k4n5
(1+(k-1)+...+(k-1)d-2))<3
It follows from(3.8) and(3.9) that the total number of vertices which can be reached
from Xiby a path of length -d, can not exceedn1-6which is--n-2ifn	
6
12
,
and this is true ifn -l k64; thus we arrived to a contradiction and this proves
our theorem.
2
To show that the order of magnitude
k2
of the lower estimate ofF3(n, k)
is best possible, consider the following graph G„: Take a complete graphGrhaving
r vertices, and connect each vertex ofGrwithr -1new points. Thus we obtain
a graphG„withr (r -1)-}r = r2= nvertices. Clearly one hask = V(G„)= 2r-2,
z
D(G„)=3andE(G„)=2r(r-1).ThusE(G„)-k	
2
.
In this examplek=2(~n-1);by slightly modifying this example we obtain that
z
F3(n,k) <
k2
(c +1)2
1
+
if k-enwhere 0 < c < 1.
To show thatF3(n, k)is of order of magnitudenfork -~)l nwhere
0 <A-::1 we have to apply a more involved construction. Let us consider a graph
G„ which has the verticesPgi;where lg--1,1--is,l-j-sand the vertices
Qghiwhere1--g<h--land1-i--s;thusn=1s2+(2)s.Suppose that the edges
of G„ are as follows:
a) Pgi;andPhilare both connected withQghifor1--g<h --<I,i,j =1, 2,...,s.
b) Qghi,is connected withQghizfor 1-
it-S,1i2S,it7z i2,1--g<h --1;
C)Qg,h,iandQa2h2 iare connected for 1g,< h

land 1 ~g2 < h2-1,
i=1,2,...,s.
Clearly
2,
E(G")
-2 ~')s2+
(1)
(SI + ~
2

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furtherv(Pgij)=s-1and
l
/

2
v(Qghi)=sfil -11

2
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and thusV(G,)=s+l+
Z
2
-2. Thus we obtain
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Z
F,Ils
+~1)S,S+l+(1+])/(181)(1-2)
21<2[jS2+IjISI+
2
s.
2

-2
By other words by choosing for Z an arbitrary fixed natural number and for s tending
to + -,we obtain an infinite sequence of pairsn,ksuch that
a
k -
l'l
andF3(n, k)

4 k2
/(1-1).
Thus for arbitrary smalla,:~-0 there exists an infinity of pairsn, ksuch thatk -;L~nand
F3 (n, k)<5

n
24a4* k2.
Let us study now the behaviour ofF3(n, k)for large values ofk.Clearly
F3(n, k) = n -1ifk -
2
because the graphGnshown on Fig. 9 has diameter 3
V(Gj = kandGnis a tree, thus it hasn -1edges; this result is best possible because
a connected graphGncannot have less thann-1edges.
Fig.9
We prove now the following
3
THEOREM4. If	
s
+l+s- 1-- ks +s -2 where s =1, 2, 3,..

2then
F3(n, k)=n+(Z
PROOF OF THEOREM4.The e ase s= 1 has been settled above. Let us consider
first the cases=2.Suppose Gnwou'd be a tree of diameter3andV(G.)=k-2 ,
and let Pibe an endpoint ofGn(such a point exists as every tree has at least two
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endpoints). Let P2denote the single point connected with Piby an edge, and let
P3,...,P,be all the other points connected with Pz;asV(G„)=kwe havel--k+1.The remainingn -k-1--k-1 --l-2points have to be connected with one of the
pointsP3,...,P,because otherwise it would be impossible to reach them from
Piby a path of length--3. But they can not be all connected with the same point
P; (3 j--l)because this point would have valencylc.LetP,andP,be two
points(Z < r < sn)such thatP,is connected with Pi,a nd P, withP; (3 -i-j 1).
Then the (unique) path from P, to P, has length 4; this contradiction shows that
F3(n, k) nfork,2.
On the other hand Fig. 10 shows a graphG„withV(G„)=kwhere
3
+ I--k2
which has diameter3and contains exactly one cycle (a triangle) and thusE(G.)=77.
This completes the proof of the fact thatF3(n, k) = nfor
3
+ 1 k2.
Note that forn=2k+1there is another extremal graphG2k+1of diameter3,
for whichV(G2h+1)=1candE(G,k+,)=2k+1,shown by Fig. 11.
Fig. 10
Now we pass to the case s3.
E(G,)-2(1+k+2(n-1-1»=n-1+2-1.
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VCG2k+1)=k,E032k+i)=2k+i, D(G2k+j)=3
Fig. II
3
LetGbe a graph withV(G„)=k 1s+,+s-l-k_s+s-2;s<2
andD(G„)=3.Let Xl,...,X,be the endpoints of G,,.Asthe remainingn-1points
all have valency--2,and at least one among them has valency k, we have

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Now ifE(G„)--n+ (2
)-1,we have nothing to prove;if howeverE(G.)n +
+(2)-1we get
l:~-k-s(s- 1) -s-1
thus1--2. LetY1,...,Y„denote those vertices ofG„which are connected with
at least oneX,(1--jl).
Clearly Yiand Y, are connected by an edge(1--i<j--v)because otherwise
there would not exist a path of length 3 connecting the Xh-s. Thus it is sufficient
to consider the case v--s,because every connected graphG„containing a complete
s+ 1-graph has at leastn -1+Z)
edges. Let us suppose therefore that v--s.
We prove first that vmss.Let the endpoint X, be connected to Y1.Let Z1,... Zr
denote all the points connected with Y, which are not endpoints of Gn.As every
point of G„ can be reached from X, by a path of length--3, if Y, is connected
and therefore, in view of s
r
withpendpoints then we havefv(Z,,)'n-p-1thus
h=1
E(G„)I (n-p-l+p+t-+1+2(n-l-r-1))=
3n23-r21
thus in case E(G,,)< n+ (Z
)-1we get
l--n-r-s(s-1).
As however eachY,has valencyk, it can be connected to at mostkof the X,-s,
and Y, only tok-r XLs;thus
(v-1)k+k-ran-r-s(s-1)
2 , we obtain v::-s -1 i.e. v--s.Thus we have cmy
to consider the casev=s.Now ifv = sthere exist inG„at least s points which
are not connected to any of theY,-sbecause these have valencies--kand thus
the total number of points connected with them is--s(k-(s-1)):n-s. LetW
be such a point.
Now clearly W has to be connected with each Xhby a path of length 3 and
therefore with eachY;by a path of length2.Let U1,...,Utbe the points connected
withW,then eachY,is connected with someU.Thus it follows
E(Gj
2
(21+s(s-1)+2s+2t+2(n-I-s-t-1)) _
=
11-2l
I+s+t+n-I-s-t-1 =n+(2)-1.
Thus F3(n, k) n+ (2)
l
.On the other hand consider the graphGnof the follow-ing structure:let us take a complete graph G5+,having s + 1 points, and connect
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each of these points except one withk - sendpoints, and the last withn -s (k-s)-
-(s+l) points. (ClearlyO--n-s(k-s)-(s +1)--k-s).
Thus we obtain a graph G„ withV(G„)= k, D (Gj = 3andE(G.)= n +(s)-1-
This completes the proof of Theorem 4.
Let us consider nowF4(n, k).Clearly
F4(n, k)=n-1ifk-fn-1.
This can be seen as follows. Fig. 12 exhibits a tree of diameter 4 showing that
F4(k2 + 1,k)=k2
Clearly if(k-1)2+ 1-- n <k2+1'we obtain a graphG„exhibitingF4(n, k) _
=n -1 byomitting from the graph on Fig. 12k2+ 1- nendpoints. We shall.
prove now
Pk+3

THEOREM 5.
1
4
F4(k2+2,k)-k2+1+25

(k=2,3,...).
P2k

P2kai
PROOF OF THEOREM 5.LetGkz+2be an ext-P2k+2
remal graph i.e. one which hask2+ 2points,
diameter 4, satisfies the conditionV(Gk2+2)=kand
hasF4(k2+ 2, k)edges.
Pzk-l

LetXI,...,X,nbe the points ofGkz+2having
valency--2,and letG,*nbe the subgraph ofGk2+2
spanned by these points. We assert that each point
X 'has the valency ~2 inG,*„too. Suppose that
~°Pk2 k+a

XIis an endpoint ofGI*,,,and that X2isthe only
Pk2-k+,,

point of Gm to which XIis connected. Clearly XI
Pkz

is connected with at least one endpoint YIof
Gkz+2because it has valency 22 inC'-k2+2,thus it
is connected with some point ofGk2+2different
from XZand this point cannot be inGn*,and
thus is an endpoint of Gk2+2.Every point ofGk2+2can be reached by supposition
from YIby a path of length4. However the number of points which can be
reached from YIby such a path is clearly
-2k-1+(k-1)2=k2
which is a contradiction. Thus in G,*neach point has valency--2.Asthe diameter
ofG,*nis--4,it follows from (1. 1) that G,*„ contains at least one point of valency
4		4	
fm-1;thus the number of edges ofG,*nexceeds(m -1) +i(~m-1).Each point
in G,*„can be connected with at mostk-2endpoints of Gk2+2thusIce+2-m+
2
+m(k-2)=m(k-1)and thereforem-k
k-
+
1
2
,k+1;thus
4		
1 "-
E(Gk2+z) k2+1+ 12
(Vm-1) --k2+1+2~k.
Thus Theorem5is proved.
pk.2
Fig.12
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Note that the statement of Theorem 5 is trivial for k--16,because it states
only what we know already that ifD(Gk2+2)=4thenG,2+2can not be a tree.
To get an upper estimate forF4(k2+2, k)- k2consider the following graph.
TakeagraphGk+swithV(Gk+s)=k, D(Gk+s)=2andE(Gk+s)=2k+6;such
a graph exists according to Theorem2ifk 8(see Fig.7withl =5).This graph
hask + 2points of valency2.Connectkout of these points withk - 2new points
each and one withk - 3new points. Thus we get a graphGnwithn = k2+ 2points,
such that V(G.)= k, D (Gn)= 4and E(G.)= k2+ k + 3.Thus
F4(k2+2,k)--k2+k+3.
§4.Some further Remarks and Unsolved Problems
First we formulate some general principles of construction which were implicitely
used above.
IfGnis a graph of diameterd,and such thatV(G,,)=k,then ifG,'is not regular,
we may construct from G„ a graphGNof orderN=n + kn- E(G,,)withV (GN)= k
and diameterd+2,by connecting each vertexPiofGnwhich has valencyv(P) <k
withk -v (P)new points. Thus
(4. 1)

Fa+2(n+kn-2Fa(n,k), k)-kn-F,i(n, k).For instance we have shown thatF2(n, n-5) = 2n-4.It follows immediately
from (4. 1) that
F4(n2-8n+8, n-5)-n2-7n+4.Notice that for each value ofd,the extremal graphs Gnwith V(G,,)= k, D(Gn)= d
and having a minimal number of edges, are trees ifkis sufficiently large,k -- Ud(n)say.
We have implicitely shown that
(4.2)

UZ(n) = n -1
U3(n)

2
(4.3)
further that for any fixed s~---3 and
(4.6)
and
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(4.4)

U4(n) _ / n -1.It can be shown that
(4.5)

Us(n)

1+~2n-3
2
n->-
U,(n)-~n
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(4.7)

U2s+ l. (n)-

2
The extremal tree of diameter 2s has a center, while the extremal tree of diameter
2s + 1 has a central edge.
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Notice that ifkdecreases by one below the critical valueU,(77),i.e. toUd(n)-1,
there is a considerable increase in the value ofF,,(n,k)ifdis even, but not ifdis
odd. As a matter of fact
FZ(n, U2(n) -1)- Fz,(n, U2, (n)) _ (2n-4)-(n -1) = n- 3
F3(2k+1, k)-F3(2k+1, k+1) _ (2k+1) -2k= I
and
F3(2k+2, k)-F3(2k+2, k+1)= (2k+3)-(2k+1)=2
further as proved by Theorem 5
1
4-
F4(k2+2,k)-F4(k2+1, k)-2lk.
The situation is similar for d>4.
We call attention to the following problems, left open in this paper:
PROBLEM1.Is the graph of Theorem 1 extremal in the sense that among all
graphs with n vertices and not containing any cycle of length 4 does it have the
maximal number of edges? (We have proved only that it is asymptotically extremal.)
We can prove the following result, which is connected with Problem 1.
THEOREM6. If G„ is a graph in which any two points are connected by a path
of length 2 and which does not contain any cycle of length 4, thenn=2k+ 1and
G„ consists of k triangles which have one common
vertex (see Fig. 13).
PROOFOFTHEOREM6. Let G„ be a graph
with the required properties. Let P, be a point of
G„ having maximal valency. If PIis connected with
all the remaining points of G„ then evidently these
have to be connected by pairs, and G„ is of the
type described in Theorem 6. Thus we may suppose
that G„ contains at least one point PZwhich is
not connected with PI.It is easy to see that in this
caseV(PZ)= V(PI).
As a matter of fact there is a point P3in G„
which is connected with both PIand P2.As there
must be a path of length 2 between PIand P3
there is a point P4which is connected with both P, and P3.As there has to be a path
of length 2 between P2and P3,there is a point PSconnected with both P2and P3,
which is clearly different fromPi, P2,.P3and P4.LetQ1,Q2,...,Qk-2be the
remaining points (besides P3and P4)which are connected with Pi.Clearly P2
and PSare not among theQ,;we havek~4becausev(P3)-4-and by supposition
P, has the maximal valency.
Now from each of the pointsQ,there is a path of length 2 to P2;thus for each
Q,(i=1, 2,...,k -2) there exists a pointR,which is connected with bothQ,andP2.
ClearlyR, R;ifijbecause otherwise G„ would contain the cyclePIQ,R,Qj.
FurtherR,is different from P3because ifR,would be identical with P3G„ would
contain the cyclePIQ,P3P4.FinallyR,is different from Psbecause otherwise G„
Fig. 13
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would contain the cyclePlQiP5P3.Thusv(P,)--kand ask=V(G,)we obtain
v(PZ)=k=v(P,).Thus any point of G„ which is not connected with Plhas the va-lencyk= v(P,).Repeating the same argument withP2instead of Piit follows that
v (Qi)= k(i= l, 2,... , k -2). As P3isnot connected withQI(because otherwiseG„
would contain the cycleP1Q1P3P4)repeating the same argument forQinstead of
Plit follows that v(P3)=k.Thus the graph G;, is regular.
Now ifV(P)=k (i=1,2,...,n)and G„ does not contain a cycle of length 4
and between any two points there is a path of length 2, then clearly if Sidenotes
the set of points connected with Pithen the sets SiandSIhave exactly one point
in common, and for any two points PiandP; (ji)there is exactly one pointP,,
such that S,, contains both Piand P,. Thus if we define the sets of points Sias lines
we obtain a finite plane geometry, withk=P-I-1points on a line, and thus having
n = PZ-I- P+1 points. But then in this geometry there would exist a one-to-one
mapping beween points and lines such that no line contains the point corresponding
to it, and such a mapping is known [5] to be impossible. This proves Theorem 6.
PROBLEM2. To determine the exact value ofFZ(n, k)fork<2,or at least
the asymptotic value ofFZ(n,[nc])with 0 < c< 2.
	
PROBLEM3. Is the lower estimate in Theorem 3 asymptotically best possible,
I
i.e. do there exist for eachd--3a sequence of graphsG„(n~)withV(G.)=k -cna-I
z
where c0 is a constant,D (Gn)= dandE(Gn) -	
ka- rncn?
PROBLEM4.Determine asymptoticallyF4(k2-I- 2, k)- k2.
Problems similar to those considered in this paper can be asked for directed
graphs. We hope to return to these problems in an other paper.
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