Lecture 5

Forms and operators

Now we introduce the main object of this course – namely forms in Hilbert spaces. They are so popular in analysis because the Lax-Milgram lemma yields properties of existence and uniqueness which are best adapted for establishing weak solutions of elliptic partial differential equations. What is more, we already have the Lumer-Phillips machinery at our disposal, which allows us to go much further and to associate holomorphic semigroups with forms.

5.1 Forms: algebraic properties

In this section we introduce forms and put together some algebraic properties. As domain we consider a vector space V over $\mathbb{K}$.

A **sesquilinear form** on V is a mapping $a\colon V \times V \to \mathbb{K}$ such that

$$a(u+v,w) = a(u,w) + a(v,w), \quad a(\lambda u,w) = \lambda a(u,w),$$
$$a(u,v+w) = a(u,v) + a(u,w), \quad a(u,\lambda v) = \overline{\lambda} a(u,v)$$

for all $u, v, w \in V$, $\lambda \in \mathbb{K}$.

If $\mathbb{K} = \mathbb{R}$, then a sesquilinear form is the same as a bilinear form. If $\mathbb{K} = \mathbb{C}$, then a is antilinear in the second variable: it is additive in the second variable but not homogeneous. Thus the form is linear in the first variable, whereas only half of the linearity conditions are fulfilled for the second variable. The form is $1\frac{1}{2}$-linear; or sesquilinear since the Latin 'sesqui' means 'one and a half'.

For simplicity we will mostly use the terminology **form** instead of sesquilinear form. A form a is called **symmetric** if

$$a(u,v) = \overline{a(v,u)} \qquad (u, v \in V),$$

and a is called **accretive** if

$$\operatorname{Re} a(u,u) \geqslant 0 \qquad (u \in V).$$

A symmetric form is also called **positive** if it is accretive.

In the following we will also use the notation

$$a(u) := a(u,u) \qquad (u \in V)$$

for the associated quadratic form.

5.1 Remarks. (a) If $\mathbb{K} = \mathbb{C}$, then each form a satisfies the **polarisation identity**

$$a(u,v) = \frac{1}{4}\big(a(u+v) - a(u-v) + \mathrm{i}a(u+\mathrm{i}v) - \mathrm{i}a(u-\mathrm{i}v)\big) \qquad (u,v \in V).$$

In particular, the form is determined by its quadratic terms. The identity also shows that a is symmetric if and only if $a(u) \in \mathbb{R}$ for all $u \in V$. This characterisation is obviously only true if $\mathbb{K} = \mathbb{C}$. So here we have a case where the choice of the field matters.

(b) If $\mathbb{K} = \mathbb{C}$, then a is positive symmetric if and only if $a(u) \in [0,\infty)$ for all $u \in V$.

Now we may again consider $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Recall that a scalar product is a symmetric form a which is definite, i.e., $a(u) > 0$ for all $u \in V \setminus \{0\}$. For Schwarz's inequality to hold we do not need definiteness. In fact, we may even consider a version which involves two symmetric forms. This will be useful later on.

Note that each form a satisfies the **parallelogram identity**

$$a(u+v) + a(u-v) = 2a(u) + 2a(v) \qquad (u,v \in V).$$

5.2 Proposition. *(Schwarz's inequality) Let $a, b\colon V \times V \to \mathbb{K}$ be two symmetric forms. Assume that $|a(u)| \leqslant b(u)$ for all $u \in V$. Then*

$$|a(u,v)| \leqslant b(u)^{1/2} b(v)^{1/2} \qquad (u,v \in V). \tag{5.1}$$

Proof. Let $u, v \in V$. In order to show (5.1) we may assume that $a(u,v) \in \mathbb{R}$ (in the complex case replace u by γu with a suitable $\gamma \in \mathbb{C}$, $|\gamma| = 1$). Then $a(u,v) = a(v,u)$ by the symmetry of a, and therefore

$$a(u,v) = \frac{1}{4}\big(a(u+v) - a(u-v)\big).$$

Hence from the hypothesis one obtains

$$|a(u,v)| \leqslant \frac{1}{4}\big(b(u+v) + b(u-v)\big) = \frac{1}{2}\big(b(u) + b(v)\big),$$

in virtue of the parallelogram identity.

Let $s > b(u)^{1/2}$, $t > b(v)^{1/2}$. Then

$$|a(u,v)| = st\left|a\Big(\frac{1}{s}u, \frac{1}{t}v\Big)\right| \leqslant st \cdot \frac{1}{2}\Big(\frac{b(u)}{s^2} + \frac{b(v)}{t^2}\Big) \leqslant st.$$

Taking the infimum over s and t we obtain (5.1). □

Finally, we introduce the adjoint form. Let $a\colon V \times V \to \mathbb{K}$ be a form. Then

$$a^*(u,v) := \overline{a(v,u)} \qquad (u,v \in V)$$

defines a form $a^*\colon V \times V \to \mathbb{K}$. Thus a is symmetric if and only if $a = a^*$. In the case of complex scalars, the forms

$$\operatorname{Re} a := \frac{1}{2}(a + a^*) \text{ and } \operatorname{Im} a := \frac{1}{2\mathrm{i}}(a - a^*)$$

are symmetric and

$$a = \operatorname{Re} a + \mathrm{i} \operatorname{Im} a.$$

We call $\operatorname{Re} a$ the **real part** and $\operatorname{Im} a$ the **imaginary part** of a. Note that $(\operatorname{Re} a)(u) = \operatorname{Re} a(u)$ and $(\operatorname{Im} a)(u) = \operatorname{Im} a(u)$ for all $u \in V$.

There is another algebraic notion – only used for the case $\mathbb{K} = \mathbb{C}$ – that will play a role in this course. A form $a\colon V \times V \to \mathbb{C}$ is **sectorial** if there exists $\theta \in [0, \pi/2)$ such that $a(u) \in \{z \in \mathbb{C} \setminus \{0\}\,;\ |\operatorname{Arg} z| \leqslant \theta\} \cup \{0\}$ for all $u \in V$. If we want to specify the angle, we say that a is **sectorial of angle** θ. It is obvious that a form $a\colon V \times V \to \mathbb{C}$ is sectorial if and only if there exists a constant $c \geqslant 0$ such that

$$|\operatorname{Im} a(u)| \leqslant c \operatorname{Re} a(u) \qquad (u \in V).$$

(The angle θ and the constant c are related by $c = \tan \theta$.)

5.2 Representation theorems

Now we consider the case where the underlying form domain is a Hilbert space V over $\mathbb{K}$. An important result is the classical representation theorem of Riesz-Fréchet: If η is a continuous linear functional on V, then there exists a unique $u \in V$ such that

$$\eta(v) = (v \,|\, u)_V \qquad (v \in V)$$

(cf. [Bre83; Théorème V.5]).

The purpose of this section is to generalise this result. First of all, in the complex case, it will be natural to consider the antidual V^* of V instead of the dual space V'. More precisely, if $\mathbb{K} = \mathbb{R}$, then $V^* = V'$ is the dual space of V, and if $\mathbb{K} = \mathbb{C}$, then we denote by V^* the space of all continuous antilinear functionals. (We recall that $\eta\colon V \to \mathbb{C}$ is called antilinear if $\eta(u + v) = \eta(u) + \eta(v)$ and $\eta(\lambda u) = \overline{\lambda}\eta(u)$ for all $u, v \in V$, $\lambda \in \mathbb{C}$.) Then V^* is a Banach space over $\mathbb{C}$ for the norm $\|\eta\|_{V'} = \sup_{\|v\|_V \leqslant 1} |\eta(v)|$. For $\eta \in V^*$ we frequently write

$$\langle \eta, v \rangle := \eta(v) \qquad (v \in V).$$

Of course, the theorem of Riesz-Fréchet can be reformulated by saying that for each $\eta \in V^*$ there exists a unique $u \in V$ such that

$$\eta(v) = (u \,|\, v)_V \qquad (v \in V).$$

We will also need the **Riesz isomorphism** $\Phi\colon V \to V^*,\ u \mapsto (u \,|\, \cdot)$. It is easy to see that Φ is linear and isometric. The Riesz-Fréchet theorem shows that Φ is surjective.

Next we derive a slight generalisation of the Riesz-Fréchet theorem, the omni-present Lax-Milgram lemma.

A form $a\colon V \times V \to \mathbb{K}$ is called **bounded** if there exists $M \geqslant 0$ such that

$$|a(u, v)| \leqslant M \|u\|_V \|v\|_V \qquad (u, v \in V). \tag{5.2}$$

It is not difficult to show that boundedness of a form is equivalent to continuity; see Exercise 5.1. The form is **coercive** if there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u) \geqslant \alpha \|u\|_V^2 \qquad (u \in V). \tag{5.3}$$

If $a\colon V \times V \to \mathbb{K}$ is a bounded form, then

$$\langle \mathcal{A}u, v\rangle := a(u,v) \qquad (u, v \in V)$$

defines a bounded operator $\mathcal{A}\colon V \to V^*$ with $\|\mathcal{A}\|_{\mathcal{L}(V,V^*)} \leqslant M$, where M is the constant from (5.2). Incidentally, each bounded operator from V to V^* is of this form. Coercivity implies that $\mathcal{A}$ is an isomorphism: this is the famous Lax-Milgram lemma.

Before stating and proving the Lax-Milgram lemma we treat the 'operator version'.

5.3 Remark. Let $A \in \mathcal{L}(V)$ be coercive, i.e.,

$$\operatorname{Re}\,(Au \,|\, u) \geqslant \alpha \|u\|_V^2 \qquad (u \in V),$$

with some $\alpha > 0$. Then, obviously, $A - \alpha I$ is accretive, and Remark 3.20 implies that $A - \alpha I$ is m-accretive. Therefore $A = \alpha I + (A - \alpha I)$ is invertible in $\mathcal{L}(V)$, and $\|A^{-1}\| \leqslant \frac{1}{\alpha}$ (see Lemma 3.16, Remark 3.17 and Lemma 3.19).

5.4 Lemma. *(Lax-Milgram) Let V be a Hilbert space, $a\colon V \times V \to \mathbb{K}$ a bounded and coercive form. Then the operator $\mathcal{A}\colon V \to V^*$ defined above is an isomorphism and $\|\mathcal{A}^{-1}\|_{\mathcal{L}(V^*,V)} \leqslant \frac{1}{\alpha}$, with $\alpha > 0$ from* (5.3).

Proof. Composing $\mathcal{A}$ with the inverse of the Riesz isomorphism $\Phi\colon V \to V^*$ we obtain an operator $\Phi^{-1}\mathcal{A} \in \mathcal{L}(V)$ satisfying

$$\operatorname{Re}\left(\Phi^{-1}\mathcal{A}u \,\middle|\, u\right) = \operatorname{Re}\,\langle \mathcal{A}u, u\rangle = \operatorname{Re} a(u,u) \geqslant \alpha\|u\|^2 \qquad (u \in V).$$

From Remark 5.3 we conclude that $\Phi^{-1}\mathcal{A}$ is invertible in $\mathcal{L}(V)$, and $\|(\Phi^{-1}\mathcal{A})^{-1}\| \leqslant \frac{1}{\alpha}$. As Φ is an isometric isomorphism we obtain the assertions. □

If the form is symmetric, then the Lax-Milgram lemma is the same as the theorem of Riesz-Fréchet. In fact, then a is an equivalent scalar product, i.e., $a(u)^{1/2}$ defines an equivalent norm on V.

5.3 Semigroups by forms, the complete case

Here we come to the heart of the course: we prove the first generation theorem. With a coercive form we associate an operator that is m-sectorial and thus yields a contractive, holomorphic semigroup. The Lumer-Phillips theorem in its holomorphic version (Theorem 3.22) characterises generators of such semigroups by sectoriality and a range condition. In concrete cases the range condition leads to a partial differential equation (mostly of elliptic type) which has to be solved. If the operator is associated with a form, then the Lax-Milgram lemma does this job, so the range condition is automatically fulfilled. At first we will explain how we associate an operator with a form.

We use the terminology "complete case" since in this lecture the form domain is a Hilbert space. After having seen a series of examples in diverse further lectures, we will also meet the "non-complete case" where the form domain is just a vector space.

Let V, H be Hilbert spaces over $\mathbb{K}$ and let $a\colon V \times V \to \mathbb{K}$ be a bounded form. Let $j \in \mathcal{L}(V, H)$ be an operator with dense range. We consider the condition that

$$u \in V,\ j(u) = 0,\ a(u) = 0 \ \text{ implies } \ u = 0. \tag{5.4}$$

Let

$$A := \big\{(x, y) \in H \times H\,;\, \exists u \in V\colon j(u) = x,\ a(u, v) = (y \,|\, j(v))\ \ (v \in V)\big\}.$$

5.5 Proposition. (a) *Assume* (5.4). *Then the relation A defined above is an operator in H. We call A the* ***operator associated with*** (a, j) *and write $A \sim (a, j)$.*

(b) *If a is accretive, then A is accretive.*

(c) *If $\mathbb{K} = \mathbb{C}$ and a is sectorial, then A is sectorial of the same angle as a.*

Proof. (a) It is easy to see that A is a subspace of $H \times H$. Let $(0, y) \in A$. We have to show that $y = 0$. By definition there exists $u \in V$ such that $j(u) = 0$ and $a(u, v) = (y \,|\, j(v))_H$ for all $v \in V$. In particular, $a(u) = 0$. Assumption (5.4) implies that $u = 0$. Hence $(y \,|\, j(v))_H = 0$ for all $v \in V$. Since j has dense range, it follows that $y = 0$.

(b), (c) If $x \in \operatorname{dom}(A)$, then there exists $u \in V$ such that $j(u) = x$ and such that $a(u, v) = (Aj(u) \,|\, j(v))$ for all $v \in V$, and then $a(u, u) = (Aj(u) \,|\, j(u)) = (Ax \,|\, x)$.

If $\operatorname{Re} a(u, u) \geqslant 0$ $(u \in V)$, then $\operatorname{Re}(Ax \,|\, x) \geqslant 0$ for all $x \in \operatorname{dom}(A)$, and this proves (b). Also, in the complex case, $\operatorname{num}(A)$ is contained in $\{a(v)\,;\, v \in V\}$, and this proves (c). $\square$

5.6 Remark. Let V, H, a, j be as above, and let $\omega \in \mathbb{R}$. Then

$$b(u, v) := a(u, v) + \omega\,(j(u) \,|\, j(v)) \qquad (u, v \in V)$$

defines a form satisfying (5.4) as well (with a replaced by b). Let B be the operator associated with (b, j).

Let $x, y \in H$. Then for all $u, v \in V$ with $j(u) = x$ we have

$$a(u, v) = (y \,|\, j(v)) \iff b(u, v) = (y + \omega x \,|\, j(v))\,.$$

This shows that

$$(x, y) \in A \iff (x, y + \omega x) \in B.$$

Therefore $B = A + \omega I$.

Now we prove the first generation theorem for forms. Note that coercivity implies (5.4).

5.7 Theorem. *(Generation theorem, complete case, part 1) Let $a\colon V \times V \to \mathbb{K}$ be bounded and coercive and let $j \in \mathcal{L}(V, H)$ have dense range. Let A be the operator associated with (a, j). Then A is m-accretive, i.e., $-A$ generates a contractive C_0-semigroup on H.*

Proof. Clearly, the hypothesis that a is coercive implies that a is accretive. Hence A is accretive by Proposition 5.5(b). In order to show that A is m-accretive we have to show the range condition $\operatorname{ran}(I+A)=H$. Define the form $b\colon V\times V\to\mathbb{K}$ by

$$b(u,v):=a(u,v)+(j(u)\,|\,j(v))\qquad (u,v\in V).$$

Then b is bounded and coercive; recall from Remark 5.6 that the operator $I+A$ is associated with (b,j).

Let $y\in H$. Then $\eta(v):=(y\,|\,j(v))_H$ defines an element $\eta\in V^*$. By the Lax-Milgram lemma there exists $u\in V$ such that

$$b(u,v)=(y\,|\,j(v))_H\qquad (v\in V).$$

This implies that $x:=j(u)\in\operatorname{dom}(A)$ and $(I+A)x=y$. □

5.8 Remark. For later use we explain that the construction presented in the proof of Theorem 5.7 yields a closed expression for the inverse of $I+A$.

In order to derive this expression we let $\mathcal{B}\colon V\to V^*$ denote the 'Lax-Milgram operator' associated with the form b used above, i.e.,

$$\langle\mathcal{B}u,v\rangle:=b(u,v)\qquad (u,v\in V).$$

Further we define $k\colon H\to V^*$, $y\mapsto(y\,|\,j(\cdot))_H$. Then

$$|\langle k(y),v\rangle|\leqslant\|y\|_H\|j(v)\|_H\leqslant\|j\|\|y\|_H\|v\|_V\qquad (y\in H,\ v\in V),$$

and this inequality implies that $k\in\mathcal{L}(H,V^*)$, $\|k\|\leqslant\|j\|$.

Now, starting with $y\in H$ we obtain (with the notation used in the proof of Theorem 5.7) $\eta=k(y)$, $u=\mathcal{B}^{-1}\eta$, $x=j(u)$. This results in $x=j\mathcal{B}^{-1}k(y)$, and using $(I+A)x=y$ and the invertibility of $I+A$ we obtain $(I+A)^{-1}=j\mathcal{B}^{-1}k$.

In the complex case one also obtains results concerning sectoriality.

5.9 Theorem. *(Generation theorem, complete case, part 2) Let $\mathbb{K}=\mathbb{C}$, let $a\colon V\times V\to\mathbb{C}$ be bounded and coercive and let $j\in\mathcal{L}(V,H)$ have dense range. Let A be the operator associated with (a,j). Then the form a is sectorial, and the operator A is m-sectorial, i.e., $-A$ generates a holomorphic C_0-semigroup on H which is contractive on a sector.*

Proof. By assumption there exist $M\geqslant 0$, $\alpha>0$ such that

$$|a(u,v)|\leqslant M\|u\|_V\|v\|_V,\quad \operatorname{Re}a(v)\geqslant\alpha\|v\|_V^2$$

for all $u,v\in V$. Thus

$$\frac{|\operatorname{Im}a(v)|}{\operatorname{Re}a(v)}\leqslant\frac{M\|v\|_V^2}{\alpha\|v\|_V^2}\leqslant\frac{M}{\alpha}$$

for all $v\in V\setminus\{0\}$. This implies that there exists $\theta\in[0,\pi/2)$ such that $|\operatorname{Arg}a(v)|\leqslant\theta$ for all $v\in V\setminus\{0\}$. Thus a is sectorial. The remaining assertions are immediate consequences of Proposition 5.5(c), Theorem 5.7 and Theorem 3.22. □

We give a first example as an illustration.

5.10 Example. Multiplication operators.

Let (Ω, μ) be a σ-finite measure space and let $m\colon \Omega \to \mathbb{C}$ be measurable such that $w(x) := \operatorname{Re} m(x) \geqslant \delta > 0$ for all $x \in \Omega$. Let $V := L_2(\Omega, w\mu)$. Assume that there exists $c > 0$ such that

$$|\operatorname{Im} m(x)| \leqslant c \operatorname{Re} m(x) \qquad (x \in \Omega).$$

Then $a(u, v) := \int u\bar{v}m \,\mathrm{d}\mu$ defines a bounded coercive form $a\colon V \times V \to \mathbb{C}$. Let $H := L_2(\Omega, \mu)$, $j(u) = u$ for all $u \in V$. Then $j \in \mathcal{L}(V, H)$, and j has dense range. Let $A \sim (a, j)$. Then one easily sees that

$$\begin{aligned} \operatorname{dom}(A) &= \{u \in V\,;\, mu \in L_2(\Omega, \mu)\}, \\ Au &= mu. \end{aligned}$$

From Section 2.2 we recall the concept of rescaling. If $-A$ is the generator of a C_0-semigroup T and $\omega \in \mathbb{R}$, then $-(A + \omega)$ generates the semigroup $(\mathrm{e}^{-\omega t}T(t))_{t\geqslant 0}$. One frequently uses the word "quasi" as prefix if something is true after rescaling. (The notation '$A + \omega$' is an abbreviation of '$A + \omega I$'; the ω stands for multiplication by the scalar ω, which is just the operator ωI.)

Let H be a Hilbert space. An operator A in H is **quasi-sectorial** if there exists $\omega \in \mathbb{R}$ such that $A + \omega$ is sectorial. The operator A is **quasi-m-sectorial** if $A + \omega$ is m-sectorial for some $\omega \in \mathbb{R}$. A **quasi-contractive holomorphic semigroup** is a holomorphic semigroup T such that $\|\mathrm{e}^{-\omega z}T(z)\| \leqslant 1$ for all $z \in \Sigma_\theta$, for some $\theta \in (0, \pi/2]$ and some $\omega \in \mathbb{R}$.

Thus A is quasi-m-sectorial if and only if $-A$ generates a quasi-contractive holomorphic C_0-semigroup.

Let $a\colon V \times V \to \mathbb{C}$ be a bounded form and let $j \in \mathcal{L}(V, H)$ have dense range. In analogy to the previous notation we could say that a is *quasi-coercive* (*with respect to* j) if there exist $\omega \in \mathbb{R}$, $\alpha > 0$ such that

$$\operatorname{Re} a(v) + \omega\|j(v)\|_H^2 \geqslant \alpha\|v\|_V^2 \qquad (v \in V), \tag{5.5}$$

but – for simplicity of notation – we prefer to call a **j-elliptic** in this case. It is obvious that (5.5) implies (5.4). Thus the operator A associated with (a, j) is defined by Proposition 5.5. Consider the form $b\colon V \times V \to \mathbb{C}$ given by

$$b(u, v) := a(u, v) + \omega\, (j(u) \,|\, j(v))_H\,.$$

Then b is bounded and coercive. Thus the operator B associated with (b, j) is m-sectorial. Remark 5.6 implies that $B = A + \omega I$, so we have proved the following more general generation theorem.

5.11 Corollary. *Let $j \in \mathcal{L}(V, H)$ have dense range, and let $a\colon V \times V \to \mathbb{C}$ be bounded and j-elliptic. Then the operator associated with (a, j) is quasi-m-sectorial.*

Later we will meet interesting situations where j is not injective. In most applications, however, j is an embedding; then we will usually suppress the letter j. The situation is described as follows. Let H and V be Hilbert spaces such that $V \overset{\mathrm{d}}{\hookrightarrow} H$. This is an abbreviation for saying that V is continuously embedded into H (abbreviated by $V \hookrightarrow H$) and that V is dense in H. Of course, that $V \hookrightarrow H$ means that $V \subseteq H$ and that for some constant $c > 0$ one has

$$\|u\|_H \leqslant c\|u\|_V \qquad (u \in V).$$

We call such a constant an **embedding constant**.

Now let $a\colon V \times V \to \mathbb{K}$ be a bounded form. We say that a is **H-elliptic** if

$$\operatorname{Re} a(v) + \omega\|v\|_H^2 \geqslant \alpha\|v\|_V^2$$

for all $v \in V$ and some $\alpha > 0$, $\omega \in \mathbb{R}$. In that case the definition of the operator A associated with a (not mentioning the given embedding of V into H) reads as follows. For $x, y \in H$ one has

$$x \in \operatorname{dom}(A),\ Ax = y \iff x \in V,\ a(x, v) = (y \,|\, v)_H \ \ (v \in V).$$

In the case $\mathbb{K} = \mathbb{C}$, this operator is quasi-m-sectorial by Corollary 5.11.

5.4 The classical Dirichlet form and other examples

Let Ω be an open set in $\mathbb{R}^n$. The classical Dirichlet form is defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ and given by

$$a(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x := \sum_{j=1}^n \int_\Omega \partial_j u \,\overline{\partial_j v} \,\mathrm{d}x.$$

It is clear that a is bounded; in fact

$$|a(u, v)| \leqslant \|\nabla u\|_{L_2(\Omega)} \|\nabla v\|_{L_2(\Omega)} \leqslant \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.$$

Here $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ and $\|\nabla u\|_{L_2(\Omega)} := \big(\sum_{j=1}^n \int_\Omega |\partial_j u|^2 \,\mathrm{d}x\big)^{1/2}$. Thus $\|u\|_{H^1(\Omega)}^2 = \|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)}^2$.

We will prove that the Dirichlet form is coercive if Ω is bounded, or more generally, if Ω **lies in a strip**, i.e., there exist $\delta > 0$ and $j_0 \in \{1, \ldots, n\}$ such that $|x_{j_0}| \leqslant \delta$ for all $x \in \Omega$.

5.12 Theorem. *(Poincaré's inequality) Assume that Ω lies in a strip. Then there exists a constant $c_{\mathrm{P}} > 0$ such that*

$$\int_\Omega |u|^2 \,\mathrm{d}x \leqslant c_{\mathrm{P}} \int_\Omega |\nabla u|^2 \,\mathrm{d}x \qquad (u \in H_0^1(\Omega)).$$

Proof. Theorem 4.12 implies that it suffices to prove the inequality for all $u \in C_{\mathrm{c}}^\infty(\Omega)$. We may assume that $j_0 = 1$; otherwise we permute the coordinates. Let $\delta > 0$ be such that

$|x_1| \leqslant \delta$ for all $x = (x_1, \ldots, x_n) \in \Omega$. Let $h \in C^1[-\delta, \delta]$, $h(-\delta) = 0$. Then by Hölder's inequality we estimate

$$\begin{aligned}\int_{-\delta}^{\delta} |h(x)|^2 \,\mathrm{d}x &= \int_{-\delta}^{\delta} \left| \int_{-\delta}^{x} h'(y) \,\mathrm{d}y \right|^2 \mathrm{d}x \\ &\leqslant \int_{-\delta}^{\delta} \left(\int_{-\delta}^{x} |h'(y)|^2 \,\mathrm{d}y \right) \left(\int_{-\delta}^{x} \mathbf{1} \,\mathrm{d}y \right) \mathrm{d}x \\ &\leqslant (2\delta)^2 \int_{-\delta}^{\delta} |h'(y)|^2 \,\mathrm{d}y.\end{aligned}$$

Let $u \in C_\mathrm{c}^\infty(\Omega)$. Applying the above estimate to $h(r) = u(r, x_2, \ldots, x_n)$ we obtain

$$\int_\Omega |u|^2 \,\mathrm{d}x \leqslant 4\delta^2 \int_\mathbb{R} \ldots \int_\mathbb{R} \int_{-\delta}^{\delta} |\partial_1 u(x_1, \ldots, x_n)|^2 \,\mathrm{d}x_1 \ldots \mathrm{d}x_n \leqslant 4\delta^2 \int_\Omega |\nabla u|^2 \,\mathrm{d}x. \qquad \square$$

In fact, we saw that the constant c_P can be chosen as d^2 where $d := 2\delta$ is an upper estimate for the width of Ω. (For bounded domains the best constant can be determined as $c_\mathrm{P} = 1/\lambda_1^\mathrm{D}$, where λ_1^D is the first eigenvalue of $-\Delta_\mathrm{D}$; we will come back to this later.) At first we revisit the Dirichlet Laplacian.

5.13 Example. The Dirichlet Laplacian.
Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $H := L_2(\Omega)$, $V := H_0^1(\Omega)$ and define $a\colon V \times V \to \mathbb{C}$ by

$$a(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x.$$

Then a is bounded and H-elliplic. Observe that $V \overset{\mathrm{d}}{\hookrightarrow} H$. Let A be the operator in H associated with a. Then

$$\begin{aligned}\operatorname{dom}(A) &= \{u \in H_0^1(\Omega);\, \Delta u \in L_2(\Omega)\}, \\ Au &= -\Delta u.\end{aligned}$$

Thus $-A = \Delta_\mathrm{D}$, the Dirichlet Laplacian as defined in Subsection 4.2.2.
If Ω lies in a strip, then a is coercive.

Proof. The inequality $a(u) + 1\,\|u\|_H^2 \geqslant \|u\|_V^2$ $(u \in V)$ – in fact an equality – shows that a is H-elliptic. Let $A \sim a$. Then for $u, f \in L_2(\Omega)$ one has $u \in \operatorname{dom}(A)$, $Au = f$ if and only if $u \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \overline{\nabla v} \,\mathrm{d}x = \int_\Omega f \overline{v} \,\mathrm{d}x \qquad (v \in H_0^1(\Omega)).$$

By Lemma 4.17 the latter is equivalent to $-\Delta u = f$ in the distributional sense (i.e., $-\int_\Omega u \overline{\Delta v} \,\mathrm{d}x = \int_\Omega f \overline{v} \,\mathrm{d}x$ for all $v \in C_\mathrm{c}^\infty(\Omega)$).

Now assume that Ω lies in a strip. Let $u \in H_0^1(\Omega)$. Then $a(u) \geqslant \frac{1}{c_\mathrm{P}} \int_\Omega |u|^2 \,\mathrm{d}x$ by Poincaré's inequality. Thus $a(u) \geqslant \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2c_\mathrm{P}} \int_\Omega |u|^2 \,\mathrm{d}x \geqslant \alpha \|u\|_{H^1(\Omega)}^2$ where $\alpha = \min\{\frac{1}{2}, \frac{1}{2c_\mathrm{P}}\}$. Thus a is coercive. $\square$

The semigroup T generated by Δ_D governs the heat equation. In fact, let $u_0 \in L_2(\Omega)$, $u(t) = T(t)u_0$ for $t \geqslant 0$. Then $u \in C([0,\infty); L_2(\Omega)) \cap C^\infty(0,\infty; L_2(\Omega))$, $u(t) \in \mathrm{dom}(\Delta_D)$ for all $t > 0$, and

$$\begin{aligned} &u'(t) = \Delta u(t), \quad u(t)|_{\partial\Omega} = 0 \qquad (t > 0),\\ &u(0) = u_0. \end{aligned}$$

(In which sense '$u(t) \in H_0^1(\Omega)$' can be expressed as '$u(t)|_{\partial\Omega} = 0$' will be explained in Lecture 7.) If we consider a body Ω (a bounded open subset of $\mathbb{R}^n$) and $u_0(x)$ as the temperature at $x \in \Omega$ at time 0, then $u(t)(x)$ is the temperature at time $t > 0$ at x. The boundary condition means that the temperature is kept at 0 at the boundary. One expects that $\lim_{t\to\infty} u(t) = 0$. This is the case as we can see in Exercise 5.2.

Finally we give an example where j is not the identity.

5.14 Example. Multiplicative perturbation of Δ_D.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set which lies in a strip. Let $m\colon \Omega \to \mathbb{C}$ be measurable such that $|m(x)| \geqslant \beta > 0$ for all $x \in \Omega$. Define the operator A in $L_2(\Omega)$ by

$$\begin{aligned} \mathrm{dom}(A) &= \bigl\{u \in L_2(\Omega);\, mu \in \mathrm{dom}(\Delta_D),\ \overline{m}\Delta(mu) \in L_2(\Omega)\bigr\},\\ Au &= -\overline{m}\Delta(mu). \end{aligned}$$

Then A is m-sectorial of angle 0.

Proof. Let $H = L_2(\Omega)$, $V = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x$, and let $j \in \mathcal{L}(V,H)$ be given by $j(v) = \frac{1}{m}v$. Then $j(V)$ is dense in $L_2(\Omega)$. In fact, let $g \in j(V)^\perp$. Then $\int_\Omega v\frac{1}{m}g\,\mathrm{d}x = 0$ for all $v \in C_c^\infty(\Omega)$. Thus $\frac{1}{m}g = 0$ by Lemma 4.5. Consequently, $g = 0$. We have shown that $j(V)^\perp = \{0\}$, i.e., $\overline{j(V)} = j(V)^{\perp\perp} = L_2(\Omega)$.

Let A be the operator in $L_2(\Omega)$ associated with (a,j). Then for $u, f \in L_2(\Omega)$ one has $u \in \mathrm{dom}(A)$ and $Au = f$ if and only if there exists $w \in H_0^1(\Omega)$ such that $\frac{w}{m} = u$ and $\int_\Omega \nabla w \cdot \overline{\nabla v}\,\mathrm{d}x = \int_\Omega f\,\overline{\left(\frac{v}{m}\right)}\,\mathrm{d}x$ for all $v \in H_0^1(\Omega)$. This is equivalent to $mu \in \mathrm{dom}(\Delta_D)$ and $-\Delta(mu) = \frac{f}{\overline{m}}$. □

Notes

The approach to forms presented here is the "French approach" following Lions [DL92]. However, we have introduced this little j following the article [AE11] (see also [AE12]). It will carry its fruits when we consider the non-complete case and also when we consider the Dirichlet-to-Neumann operator.

The Lax-Milgram lemma was proved in 1954 and is a daily tool for establishing weak solutions since then. It is an interesting part of the history of ideas that Hilbert considered bilinear forms to treat integral equations in his famous papers in the beginning of the 20th century. His ideas led his students to develop the notion of operators in Hilbert spaces. Since then we consider operators as the central objects and formulate physical and other problems with the help of operators. In the 1950's, form methods were developed to solve equations defined by operators. Forms are most appropriate for numerical treatments. The reason is that a form $a\colon V \times V \to \mathbb{C}$ can easily be operators there might be only few invariant subspaces. The method of finite elements is based on such restrictions.

Exercises

5.1 Let V be a Hilbert space, and let $a\colon V \times V \to \mathbb{K}$ be a sesquilinear form. Show that a is bounded if and only if a is continuous.

5.2 (a) Let V, H be Hilbert spaces over $\mathbb{C}$, $j \in \mathcal{L}(V, H)$ with dense range, $a\colon V \times V \to \mathbb{C}$ bounded and coercive. Let $A \sim (a, j)$ and let T be the semigroup generated by $-A$. Show that there exists $\varepsilon > 0$ such that $\|T(t)\| \leqslant \mathrm{e}^{-\varepsilon t}$ for all $t \geqslant 0$. (Hint: Show that $b(u,v) = a(u,v) - \varepsilon\,(j(u) \,|\, j(v))_H$ defines a coercive form if $\varepsilon > 0$ is small enough.)

(b) Let $\Omega \subseteq \mathbb{R}^n$ be an open set which lies in a strip. Show that

$$\big\|\mathrm{e}^{t\Delta_{\mathrm{D}}}\big\|_{\mathcal{L}(L_2(\Omega))} \leqslant \mathrm{e}^{-\varepsilon t} \qquad (t \geqslant 0)$$

for some $\varepsilon > 0$. Express $\varepsilon > 0$ in terms of the width of Ω.

5.3 Let $-\infty < a < b < \infty$. In the following we will always use the continuous representative for a function in $H^1(a,b)$; recall Theorem 4.9 for the inclusion $H^1(a,b) \subseteq C[a,b]$.

(a) Show that each $u \in H^1(a,b)$ is Hölder continuous of index 1/2, i.e., $|u(t) - u(s)| \leqslant c|t-s|^{1/2}$ for some $c > 0$.

(b) Show that the embedding $H^1(a,b) \hookrightarrow C[a,b]$ is compact, i.e., if $(u_n)_{n\in\mathbb{N}}$ in $H^1(a,b)$, $\|u_n\|_{H^1(a,b)} \leqslant c$ for all $n \in \mathbb{N}$, then $(u_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

(c) Let $H^2(a,b) := \{u \in H^1(a,b);\, u' \in H^1(a,b)\}$. Then $u'' = (u')'$ is defined for all $u \in H^2(a,b)$. Show that $H^2(a,b) \hookrightarrow C^1[a,b]$ if $C^1[a,b]$ carries the norm $\|u\|_{C^1} = \|u\|_{C[a,b]} + \|u'\|_{C[a,b]}$.

5.4 Let $-\infty < a < b < \infty$ and $\alpha, \beta > 0$. Define the operator A in $L_2(a,b)$ by

$$\begin{aligned} \mathrm{dom}(A) &= \big\{u \in H^2(a,b);\ -u'(a) + \alpha u(a) = 0,\ u'(b) + \beta u(b) = 0\big\}, \\ Au &= -u''. \end{aligned}$$

(See Exercise 5.3(c) for the definition of $H^2(a,b)$ and the existence of $u'(a)$ and $u'(b)$.)

(a) Show that A is m-sectorial. (Hint: Consider the form given by

$$a(u,v) = \int_a^b u'\overline{v'}\,\mathrm{d}x + \alpha u(a)\overline{v(a)} + \beta u(b)\overline{v(b)} \qquad (u, v \in H^1(a,b)).)$$

(b) Show that $\big\|\mathrm{e}^{-tA}\big\|_{\mathcal{L}(L_2(a,b))} \leqslant \mathrm{e}^{-\varepsilon t}$ $(t \geqslant 0)$ for some $\varepsilon > 0$.

References

[AE11] W. Arendt and A. F. M. ter Elst: Sectorial forms and degenerate differential operators. *J. Operator Theory* **67**, 33–72 (2011).

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