On Obfuscating Point Functions

A recent publication of Hoeteck Wee

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Goals of today's lecture

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- How to construct the obfuscator.

We as well will hopefully learn more about *one-way functions* and *permutations* as our obfuscator is based on a probabilistic hash function.

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- One of the few instances in cryptography where a random oracle can be replaced by a cryptographic construction.

Overview

Introduction

Short Repetition: Obfuscation

Feasibility of such obfuscators

Constructing Obfuscators for Point Functions

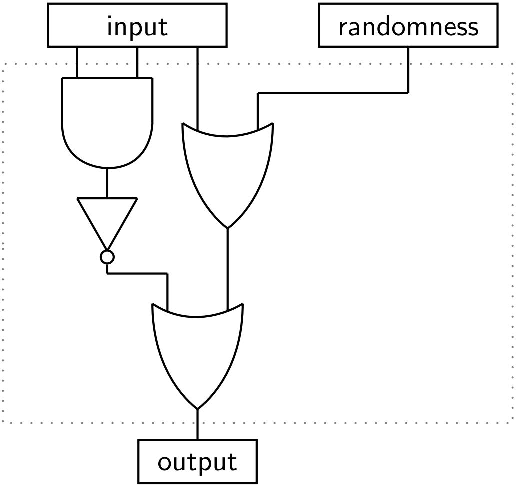
A special one-way permutation

A statistically collision-resistant hash function

Constructing the obfuscator

Short Repetition: (probabilistic) circuits

Circuit: standard boolean circuit with AND, OR and NOT gates.



Short Repetition: Obfuscation (intuitive)

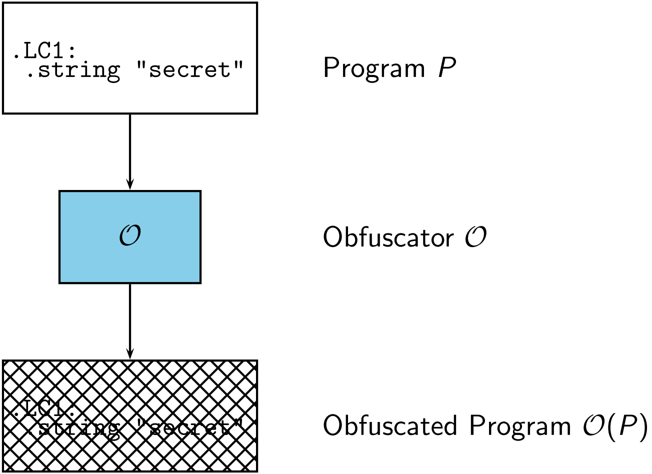


Figure: Idealized obfuscation

Short Repetition: Obfuscation (more formal)

Definition

A probabilistic polynomial-time algorithm $\mathcal{O}$ is an obfuscator for the family of circuits $\mathcal{C}_n$ (where $\mathcal{C}_n$ is the set of circuits in $\mathcal{C}$ that take inputs of length n) if the following three conditions hold:

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- approximate functionality
- polynomial slowdown
- "virtual black-box" property

Obfuscation: “Virtual black-box” property (I)

For every adversary A, there exists a simulator S, that only has black-box access to the circuit C. It should be indistinguishable whether A runs, or the simulator.

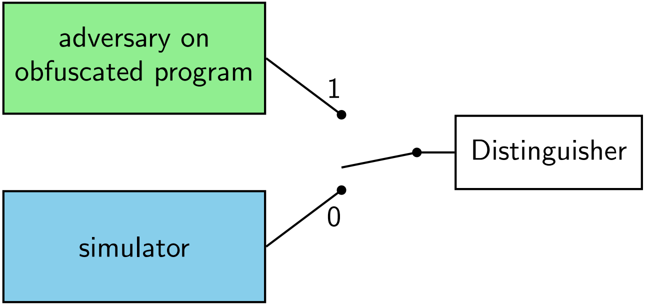

Figure: Indistinguishability of simulator and adversary

“Virtual black-box” property with a weak simulator (I)

Definition

An obfuscator $\mathcal{O}$ for a set of circuits $\mathcal{C}_n$ is (K, s, ϵ)-virtual black-box if it satifies the following: for any probabilistic circuit A of size s, there exists a probabilistic circuit S_A of size K such that for all circuits $C \in \mathcal{C}_n$,

$$\left| \Pr[A(\mathcal{O}(C)) = 1] - \Pr[S_A^C = 1] \right| < \epsilon$$

“Virtual black-box” property with a weak simulator (II)

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- Probability not negligible as in last week's definition.
- Adversary's circuit is bounded (by s).

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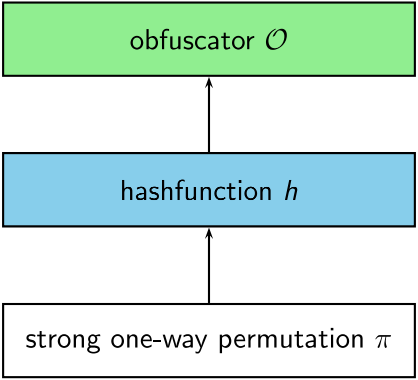
- Yes and no.
- Barak et al. showed in their paper that efficient program obfuscators do not exist.
- However this proof only holds if we need to obfuscate an arbitrary program.
- So we can still hope to find an efficient obfuscator for restricted but nonethless useful classes of programs.

Constructing Obfuscators: The big picture



Assumption: A special one-way permutation exists

- strong one-way permutation $\pi : \{0,1\}^n \rightarrow \{0,1\}^n$ wherein any polynomial-sized circuit A of size $p(n)$ inverts the permutation on at most a polynomial number of inputs $q(n)$.

$$\Pr_{x \in U_n}[A_n(\pi(x)) = x] \leq q(n)/2^n.$$

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- Standard cryptographic assumptions assert hardness with respect to work, i.e. time over success probability.

A statistically collision-resistant hash function

Construction

Let $\pi : \{0,1\}^n \to \{0,1\}^n$ *be a permutation. We define a (public-coin) probabilistic function* $h : \{0,1\}^n \times \{0,1\}^{3n^2} \to \{0,1\}^{3n^2+3n}$ *as follows:*

$$h(x; \tau_1, \ldots, \tau_{3n}) := (\tau_1, \ldots, \tau_{3n}, \langle x, \tau_1 \rangle, \langle \pi(x), \tau_2 \rangle, \ldots, \langle \pi^{3n-1}(x), \tau_{3n} \rangle)$$

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Constructing the obfuscator: Some (possible) approaches

Random oracle

Apply a random oracle to the secret value and store output. Works. Approach taken by Lynn et al. however not very ingenious as RO has already obfuscator properties.

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The construction of Wee

Won't need a random oracle, but the hash function from the previous slide.

Obfuscators for point functions exist

Theorem

Suppose there exists a $(poly(n, 1/\epsilon)s, \frac{\epsilon K}{16n} \cdot \frac{1}{2^n})$-one-way permutation, then there exists a public-coin obfuscator for the family of point functions $\{I_x\}_{x \in \{0,1\}^n}$ which is $(Kpoly(n), s, \epsilon)$-virtual black-box.

The obfuscator for point functions

Given

- π a $(poly(n, 1/\epsilon)s, \frac{\epsilon K}{16n} \cdot \frac{1}{2^n})$-one-way permutation.
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The obfuscator $\mathcal{O}(I_x; R)$

1. store $h(x; R)$ (which contains R as a substring)
2. on input y check whether $h(y; R) = h(x; R)$. If so, output 1, 0 otherwise.

The simulator for point functions (I)

Non-uniform advice

Simulator S_A of size $Kpoly(n)$, that has non-uniform advice L about adversary hardwired into it:

$$L = \{x \in \{0,1\}^n : |\Pr_R[A(h(x;R)) = 1] - \Pr[A(U_{3n^2+3n}) = 1]| \geq \epsilon\}$$

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What's this set?

The set of all point functions (which are completely determined by x), for which A is able to distinguish the obfuscation of the point function from a random string.

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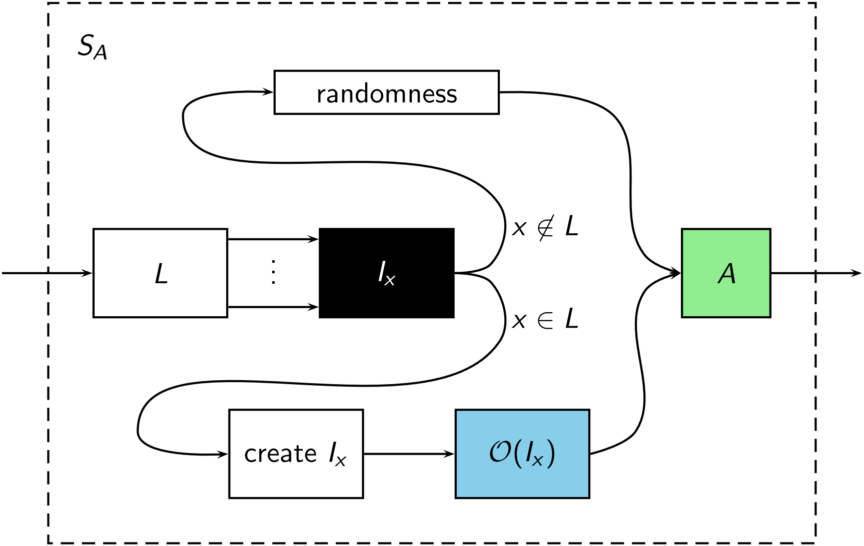
3. If $x \notin L$:
   3.1 Output a random string of correct length to the adversary.

The simulator for point functions (III)



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- As $|L| \leq K - 1$ we can hardwire all the advice L into the simulator.
- If $x \in L$, S_A computes a correct obfuscation of the point function I_x. A can by no means distinguish this from an obfuscation of the same point function.
- Else if $x \notin L$, the adversary will succeed with probability less than ϵ by the definition of L.

Excursus: Hybrid argument - Key idea

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If m instances of a problem are easy, one instance of the same problem can not be hard.

- Given a distinguisher A for *m instances* of a problem, which is successful with probability ϵ.
- We can then create a distinguisher A' based on A for *one instance* of the same problem, with advantage ϵ/m.

Excursus: Hybrid argument - Example (semantic security)

Given: Distinguisher A with advantage ϵ

which is able to distinguish m encryptions

$$c_1, \ldots, c_m$$

from random values

$$r_1, \ldots, r_m.$$

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One can create: Distinguisher for only one sample α

1. select a random i for $1 \leq i \leq m$.
2. Create the distribution

$$c_1, \ldots, c_{i-1}, \alpha, r_{i+1}, \ldots, r_m$$

3. Let A run on this distribution. We have advantage ϵ / m.

Proof of $|L| \leq K - 1$: Overview / Getting started

Contradiction if $|L| \geq K$

We can construct a circuit that inverts the one-way function on $\frac{\epsilon K}{16n}$ inputs. In the following we set $|L| = K$.

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Getting rid of the absolute value

There exists a subset L' of L of size at least $K/2$ such that for all $x \in L'$:

$$\Pr_R[A(h(x;R)) = 1] - \Pr[A(U_{3n^2+3n}) = 1] \geq \epsilon$$

Proof of $|L| \leq K - 1$: Hybrid argument

for all $x \in L'$:

$$\mathrm{Pr}_R[A(h(x;R)) = 1] - \mathrm{Pr}[A(U_{3n^2+3n}) = 1] \geq \epsilon$$

The hybrid argument

yields an j, for which we can distinguish $\langle \pi^{j-1}(x), \tau_j \rangle$ from a random bit, with advantage (average over $x \in L'$)

$$\geq 1/2 + \epsilon/3n$$

Proof of $|L| \leq K - 1$: Averaging argument

For $x \in L'$ we have advantage (average)

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Now there must be at least a $\epsilon/6n$ fraction of x in L', for which we have advantage

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To see this, examine the worst case: Nearly all x with probability $1/2 + \epsilon/6n - \delta$ ($\delta \to 0$) and very few with probability 1:

$$\frac{\epsilon}{6n} \cdot 1 + \left(1 - \frac{\epsilon}{6n}\right) \cdot \left(\frac{1}{2} + \frac{\epsilon}{6n} - \delta\right)$$
$$= \frac{1}{2} + \frac{\epsilon}{3n} + \delta\frac{\epsilon}{6n} - \frac{\epsilon}{12n} - \frac{\epsilon^2}{36n^2} - \delta \leq \frac{1}{2} + \frac{\epsilon}{3n}$$

In the worst case we get a smaller average than we should get.

Proof of $|L| \leq K - 1$: Putting it all together

- For a $\epsilon/6n$ fraction of L' we have advantage $\geq 1/2 + \epsilon/6n$.
- An older theorem of Goldreich and Levin proves that we can invert $\pi(x)$ (i.e. find the x) with probability $3/4$, if we can guess $\langle \pi(x), \tau \rangle$ with some advantage.
- $|L'| \geq K/2$.

So the total advantage is:

$$\frac{\epsilon}{6n} \cdot \frac{K}{2} \cdot \frac{3}{4} = \frac{\epsilon K}{16n}$$

I hope you don't feel this way . . .

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Take home message

Obfuscation of point functions is possible, although with some assumptions/simplifications.