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n-point sets and graphs of functions

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ABSTRACT

We prove that, for every $n \geqslant 2$, there exists an n-point set (a plane set which hits every line in exactly n points) that is homeomorphic to the graph of a function from $\mathbb{R}$ to $\mathbb{R}$; for $n \geqslant 4$, there exist both 0-dimensional and 1-dimensional examples. This raises the question (which we do not answer) of whether n-point sets for different n's could be homeomorphic.

1. Introduction

An n-point set (a plane set which hits each line in exactly n points [4]) and the graph of a function from $\mathbb{R}$ to $\mathbb{R}$ (a plane set which hits each vertical line in exactly one point) have simple and somewhat similar definitions, but clearly no set can be both. The question considered here is whether an n-point set can be homeomorphic to such a graph. It is known [3] that each 2-point set with two points removed is homeomorphic to such a graph, while there exists [3] a 2-point set which is not. It will be shown that for every $n \geqslant 2$, there exists an n-point set which is homeomorphic to the graph of a function from $\mathbb{R}$ to $\mathbb{R}$.

The following theorem contains the main results:

Main Theorem. *For each $n \geqslant 2$, there exists an n-point set which is homeomorphic to the graph of a function from $\mathbb{R}$ to $\mathbb{R}$. For each $n \geqslant 4$, there exist both zero-dimensional and one-dimensional examples.*

The $n = 2$ case (Theorem 1) contains the basic constructions; Theorem 2 ($n \geqslant 3$, zero-dimensional) extends them; and Theorem 3 extends again to $n \geqslant 4$, one-dimensional. (See [2] for dimensions.)

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2. Preliminaries

The following lemma is inspired by the first two paragraphs of the proof of the corollary of [3, p. 5]; it is a crucial tool for the constructions here. (ℓ represents a line, and ℓ_{AB} is the line determined by A and B.)

Lemma 1. *Let $\{A, B\}$ be distinct points of $\mathbb{R}^2$ with A and B on a horizontal line other than the x-axis. Then there is an embedding $\eta : \mathbb{R}^2 \setminus \ell_{AB} \to \mathbb{R}^2$ such that*

(i) *If ℓ is a line through A and $\ell \neq \ell_{AB}$, then $\eta(\ell \setminus \{A\})$ is contained in the vertical line through the point $\ell \cap \{x\text{-axis}\}$.*
(ii) *η is the identity on the x-axis.*
(iii) *The image of η is $\mathbb{R}^2 \setminus \{\text{one line}\}$.*
(iv) *η and η^{-1} preserve collinearity ('preserve lines', allowing one point on a line to be missing from the domain or image).*

Proof. Well-known properties of the projective geometry of the projective plane will be used, with the representation $\mathbb{P}^2 = \mathbb{R}^2 \cup \mathbb{P}^1$, where $\mathbb{R}^2$ is the ordinary Euclidean plane and the 'ideal points' $\mathbb{P}^1$ are the pairs of antipodal points of the unit circle. $\mathbb{P}^1$ is a line of $\mathbb{P}^2$; each other line consists of an ordinary line $ax + by = c$ of $\mathbb{R}^2$ plus one 'ideal point', the intersection of $ax + by = 0$ with the unit circle; for example, $\pm(0, 1)$ is the point of intersection of the y-axis with the 'line at infinity' $\mathbb{P}^1$. Let C and D be distinct points of the x-axis of $\mathbb{R}^2 \subset \mathbb{P}^2$. Each of the ordered quadruples $\{A, B, C, D\}$ and $\{\pm(0, 1), \pm(1/\sqrt{2}, 1/\sqrt{2}), C, D\}$ of $\mathbb{P}^2$ has no three points collinear, so there exists a unique line-preserving homeomorphism $\eta : \mathbb{P}^2 \to \mathbb{P}^2$ taking the first ordered quadruple to the second. η is the identity on the x-axis, since C, D, and $\pm(1, 0)$ are fixed; $\pm(1, 0)$ is the intersection of the line through A and B with the line through C and D. □

Lemma 2. *Let X be a metric space and let $P \subset X$ be a countably infinite subset each of whose points is open, and which has no accumulation point in X. If $F \subset P$ is finite, then X is homeomorphic to $X \setminus F$ by a homeomorphism which is the identity on $X \setminus P$.*

Proof. Let $P = \{p_k\colon 0 \leqslant k < \omega\}$ and $F = \{p_0, p_1, \ldots, p_{n-1}\}$. The map $p_k \mapsto p_{k+n}$, for $p_k \in P$, and the identity on $X \setminus P$, is the desired homeomorphism. □

The construction of n-point sets is by transfinite induction; the following will be the basic construction method used. This lemma is essentially Theorem 5.2 of [1], restated so that some additional conditions can be added to get n-point sets with special properties.

Lemma 3. *Let n and m be integers with $n \geqslant m+2$, and let M be a partial m-point set; $M = \emptyset$ and $m = 0$ is allowed. Let $\{\ell_\alpha\colon 0 \leqslant \alpha < \mathbf{c}\}$ be a well-ordering of all the lines of $\mathbb{R}^2$; let $S_0 \subset \ell_0$ satisfy $|\ell_0 \cap (M \cup S_0)| = n$.*

(i) *By transfinite induction, subsets $\{S_\alpha\colon 0 \leqslant \alpha < \mathbf{c}\}$ of $\mathbb{R}^2 \setminus M$ exist satisfying*
   (Notation: $T_\alpha = \bigcup\{S_\lambda\colon \lambda < \alpha\}$ and $\mathcal{L}_\alpha = \bigcup\{\ell_{ab}\colon a, b \in T_\alpha,\ a \neq b,\ \ell_{ab} \neq \ell_\alpha\}$)
   (1) *$|S_\alpha| < \mathbf{c}$, and if $\lambda < \alpha$ then $S_\lambda \subset S_\alpha$.*
   (2) *$M \cup S_\alpha$ is a partial n-point set (hits each line in $\leqslant n$ points).*
   (3) *$|\ell_\alpha \cap (M \cup S_\alpha)| = n$.*
   (4) *$S_\alpha \setminus T_\alpha \subset \ell_\alpha \setminus \mathcal{L}_\alpha$; in fact, the set of points of ℓ_α which are* not *eligible to be chosen for $S_\alpha \setminus T_\alpha$ has cardinality less than $\mathbf{c}$.*
   (5) *if $\alpha < \beta$, then $|S_\alpha \cap \ell_\beta| \leqslant 2$.*
(ii) *For any such collection $\{S_\alpha\}$ the set $S = M \cup \{S_\alpha\colon 0 \leqslant \alpha < \mathbf{c}\}$ is an n-point set.*

Proof. This is the same as Theorem 5.2 of [1], where conditions (4) and (5) are implicit. □

Two sorts of extra conditions will be imposed to end up with n-point sets with special properties. First, the lines $\{\ell_k\colon 0 \leqslant k < \omega\}$ at the start of the well-ordering will be specified. Second, if $\{F_\alpha\}$ is a collection of subsets of $\mathbb{R}^2$ satisfying $|F_\alpha \cap (\ell_\alpha \setminus M)| = \mathbf{c}$, then the extra condition $S_\alpha \setminus T_\alpha \subset F_\alpha$ can be added to (5). If one set F works as every F_α and $S_0 \subset F$, then the resulting n-point set satisfies $S \subset M \cup F$.

3. Results and proofs

All the constructions will start out with the following.

Standard beginning. For $i = 1, 2, \ldots$, let A_i be the open annulus bounded by the two circles with center the origin and radii i and $i + 1$; let p_0 be the point $(0, 1/2)$, which is not in any of the A_i's. Note that each line contains $\mathbf{c}$ points of all except possibly finitely many of the A_i's. Let $U = \bigcup\{A_i\colon\ i \text{ is odd}\}$. Let ℓ_0 be the horizontal line through p_0; let $\{\ell_k\colon\ 1 \leqslant k < \omega\}$ enumerate all lines through p_0 which hit the x-axis in integer points $(z, 0)$, $z \in \mathbb{Z}$. Extend $\{\ell_k\colon\ 0 \leqslant k < \omega\}$ to a well-ordering $\{\ell_\alpha\colon\ 0 \leqslant \alpha < \mathbf{c}\}$ of all the lines in the plane. Let M be a partial m-point set ($M = \emptyset$, $m = 0$ is allowed) such that $M \cap \ell_k = \emptyset$,

$0 \leqslant k < \omega$. Let $S_0 \subset \ell_0$ be a set containing p_0 such that $|\ell_0 \cap \bigcup S_0| = n$, $\ell_0 \cap (S_0 \setminus \{p_0\}) \subset U$, and no two points of $\ell_0 \cap S_0$ are in the same A_i. Apply the induction step of Lemma 3 to construct $\{S_k\colon 1 \leqslant k < \omega\}$, subject to two extra conditions: (i) all points chosen are in U, and (ii) no two points chosen are in the same A_i. Let P be the countable set of points chosen by the end of this step; each point of P is isolated (in P).

Theorem 1. *There exists a* 2*-point set that is homeomorphic to the graph of a function from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Let $M = \emptyset$, $m = 0$, $n = 2$. Apply the standard beginning. Let $V = \bigcup\{A_i\colon i \text{ is even}\}$, and use Lemma 3 to extend the construction to $\{S_\alpha\colon \omega \leqslant \alpha < \mathbf{c}\}$, subject to the extra condition that all points added are in V. Let $S = \bigcup\{S_\alpha\colon 0 \leqslant \alpha < \mathbf{c}\}$.

Now apply Lemma 1 with $A = p_0$ and $B \in S_0 \setminus \{p_0\}$; this gives a homeomorphism of $S \setminus S_0$ to the graph of a function. Apply Lemma 2 with $F = S_0$. $\square$

Theorem 2. *For each* $n \geqslant 3$ *there exists a* 0*-dimensional* n*-point set homeomorphic to the graph of a function from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Let $M = \emptyset$, $m = 0$, and apply the standard beginning.

At this point, simply repeating the construction of Theorem 1 would result in an n-point set homeomorphic to a set which hits each vertical line in exactly $n-1$ points; to be able to convert this to the graph of a function requires more care. First, to make sure that the end result will be 0-dimensional, let $\mathcal{C}$ be a countable collection of circles, each missing P, such that $\{\operatorname{int} C\colon C \in \mathcal{C}\}$ is a basis for the topology of $\mathbb{R}^2$. For i even, let $V_i = A_i \setminus \bigcup \mathcal{C}$; and let $V = \bigcup\{V_i\colon i \text{ is even}\}$. Notice that each line in the plane contains $\mathbf{c}$ points of all except finitely many of the V_i's. All future points will be chosen in V, which will force the resulting set to be 0-dimensional.

For $j = 1, 2, \ldots, n-2$, let $V^j = V_{2j}$, and let $V^{n-1} = \bigcup\{V_i\colon i \text{ is even and } i \geqslant 2(n-1)\}$. Note $V = V^1 \cup \cdots \cup V^{n-1}$, and for $\alpha \geqslant \omega$, if $p_0 \in \ell_\alpha$ then ℓ_α hits each V^j in $\mathbf{c}$ points. Now use Lemma 3 to extend $\{S_k\colon 0 \leqslant k < \omega\}$ to all of $\{S_\alpha\colon 0 \leqslant \alpha < \mathbf{c}\}$ choosing points meeting two extra conditions: (i) all points chosen are in V, and (ii) if $p_0 \in \ell_\alpha$, then ℓ_α contains at most one point in each V^j, $j = 1, 2, \ldots, n-1$. Condition (5) of Lemma 3 makes (ii) possible. Note that condition (ii) results in $|(\ell_\alpha \cap S_\alpha) \cap V^j| = 1$, for each $\alpha \geqslant \omega$ such that $p_0 \in \ell_\alpha$, and for each V^j.

Let S be the resulting n-point set; each line through p_0 hits S in $n-1$ additional points, either all in P or one in each V^j. Lemma 2 gives a homeomorphism $S \to S \setminus S_0$ which leaves all points of $S \setminus P$ fixed; then Lemma 1 gives an embedding of $S \setminus S_0$ into $\mathbb{R}^2$. Let $g : S \to \mathbb{R}^2$ be the resulting embedding. Each vertical line hits $g(S)$ in exactly $n-1$ points.

Fix an integer $k \in \mathbb{Z}$, let $(k, k+1)$ and $[k, k+1)$ be the intervals in the x-axis, and let H be the vertical strip $H = (k, k+1) \times \mathbb{R} \subset \mathbb{R}^2$; also let $\widetilde{H} = [k, k+1) \times \mathbb{R}$. Then $g^{-1}(H) \subset V$ is a (relative) open (and closed) subset of S; call it W. For $j = 1, 2, \ldots, n-1$, let $W^j = W \cap V^j$; then every line through p_0 which hits W contains exactly one point of each W^j, and $g|_{W^j} : W^j \to \mathbb{R}^2$ is an embedding of W^j whose image is the graph of a function with domain $(k, k+1)$.

The set $g^{-1}(\{k\} \times \mathbb{R})$ consists of $n-1$ isolated points of S; call them $\{q_1, q_2, \ldots, q_{n-1}\}$ and let $\widetilde{W}^j = \{q_j\} \cup W^j$. Each $g|_{\widetilde{W}^j}$ is an embedding, and each of the sets $g(\widetilde{W}^j)$ is the graph of a function on $[k, k+1)$ and $g(\widetilde{W}^1), \ldots, g(\widetilde{W}^{n-1})$ are $n-1$ pairwise disjoint relative open sets of $g(S)$; 'horizontal sliding' will be used to turn them into the graph of a single function on $[k, k+1)$.

The points $\{k + \frac{j}{n-1}\colon 1 \leqslant j \leqslant n-2\}$ partition $[k, k+1)$ into $n-1$ subintervals, $\{[k + \frac{j-1}{n-1}, k + \frac{j}{n-1})\colon 1 \leqslant j \leqslant n-1\}$. For $j = 1, \ldots, n-1$, let $\theta^j : [k, k+1) \to [k + \frac{j-1}{n-1}, k + \frac{j}{n-1})$ be a homeomorphism. Define an embedding $h : g^{-1}(\widetilde{H}) = \widetilde{W}^1 \cup \cdots \cup \widetilde{W}^{n-1} \to \widetilde{H}$ by: $h|_{\widetilde{W}^j}$ is g followed by the map $(x, y) \mapsto (\theta^j(x), y)$. Each vertical line in $\widetilde{H}$ hits $h(g^{-1}(\widetilde{H}))$ in exactly one point. Repeating this construction of h for every $[k, k+1)$, $k \in \mathbb{Z}$, gives an embedding $h : S \to \mathbb{R}^2$ such that $h(S)$ is the graph of a function. $\square$

Theorem 3. *For each* $n \geqslant 4$ *there exists a* 1*-dimensional* n*-point set homeomorphic to the graph of a function from* $\mathbb{R}$ *to* $\mathbb{R}$.

Proof. Let $M \subset A_2$ be a closed subarc of a circle centered at p_0 such that $M \cap \ell_k = \emptyset$, $0 \leqslant k < \omega$; M is a partial 2-point set. Apply the standard beginning using M and $m = 2$. Let $C(M) = \bigcup\{\ell\colon p_0 \in \ell \text{ and } \ell \cap M \neq \emptyset\}$, and let $V^1 = A_2 \setminus C(M)$. For $j = 2, 3, \ldots, n-2$, let $V^j = A_{2j}$; and let $V^{n-1} = \bigcup\{A_i\colon i \text{ is even and } i \geqslant 2(n-1)\}$. Let $V = V^1 \cup V^2 \cup \cdots \cup V^{n-1}$. For $\alpha \geqslant \omega$, ℓ_α hits V in $\mathbf{c}$ points; and if $p_0 \in \ell_\alpha$ then ℓ_α hits each of $V^2, V^3, \ldots, V^{n-1}$ in $\mathbf{c}$ points, and either hits V^1 in $\mathbf{c}$ points or hits M in exactly one point.

Use Lemma 3 to extend $\{S_k\colon 0 \leqslant k < \omega\}$ to $\{S_\alpha\colon 0 \leqslant \alpha < \mathbf{c}\}$, choosing points that meet two extra conditions: (i) all points chosen are in V, and (ii) if $p_0 \in \ell_\alpha$, then $\ell_\alpha \cap S_\alpha$ contains at most one point in each V^j. In the resulting n-point space $S = M \cup (\bigcup\{S_\alpha\colon 0 \leqslant \alpha < \mathbf{c}\})$, if $p_0 \in \ell$, then there are three possibilities for $(S \cap \ell) \setminus \{p_0\}$: it consists of (a) $n-1$ points of U, or (b) one point in each of $V^1, V^2, \ldots, V^{n-1}$, or (c) one point in each of $M, V^2, \ldots, V^{n-1}$.

Define $g : S \to \mathbb{R}^2$ as in Theorem 2; $g(M)$ is an arc which is contained in one $H = [k, k+1) \times \mathbb{R}$. Use the same 'horizontal shifting' as in Theorem 2 (0-dimensionality was not used) to get an embedding $h : S \to \mathbb{R}^2$ such that $h(S)$ is the graph of a function. $\square$

4. Questions

We have shown that n-point sets for different n's can share the same topological property—that of being homeomorphic to the graph of a function.

Question 1. Can a 2-point set and a 3-point set be homeomorphic? More generally, for which (if any) $n, m \geqslant 2$, $n \neq m$, do there exist homeomorphic n-point and m-point sets?

One might try to answer this question by considering a 2-point set S_2 and a 3-point set S_3, each homeomorphic to the graph of a function, and projecting one graph vertically onto the other; however, this is unlikely to be continuous in general. Hence a weaker question is of interest.

Question 2. For which (if any) $n, m \geqslant 2$, $n \neq m$, do there exist an n-point set S_n and m-point set S_m and a continuous bijection $S_n \mapsto S_m$?

By [3] there exist a 2-point set which is not homeomorphic to the graph of a function from $\mathbb{R}$ to $\mathbb{R}$; and by [1] for $n \geqslant 4$ there exist n-point sets which contain circles; clearly these cannot be homeomorphic to graphs.

Question 3. Does there exist a 3-point set which is not homeomorphic to the graph of a function $\mathbb{R} \to \mathbb{R}$?

Our constructions rely heavily on isolated points. A more complicated use of Lemma 1 results in a *dense* 2-point set which is homeomorphic to the graph of a function.

Question 4. For which (if any) $n \geqslant 3$ do there exist n-point sets which are dense in $\mathbb{R}^2$ and are homeomorphic to the graph of a function from $\mathbb{R}$ to $\mathbb{R}$?, or '…which contain no isolated points and are homeomorphic …', or '…contain only finitely many isolated points and …'.

References

[1] K. Bouhjar, J.J. Dijkstra, J. van Mill, Three point sets, Topology Appl. 112 (2) (2001) 215–227.
[2] D.L. Fearnley, L. Fearnley, J.W. Lamoreaux, On the dimension of n-point sets, Topology Appl. 129 (2003) 15–18.
[3] A. Fedeli, G. Gruenhage, A. Le Donne, Connectedness properties of special subsets of the plane, Topology Appl. 117 (2002) 1–7.
[4] S. Mazurkiewicz, Sur un ensemble plan qui a avec chaque droite deux et seulemnt deux points communs (1914), in: Traveaux de Topologie et ses Applications, PWN, Warsaw, 1969, pp. 46–47.