Homework 1 – Partial Differential Equations

(ODE "Review")

Give careful write-ups of the following problems. Your submitted work must include the problems being worked in the order given below, with the solutions written in a neat and organized fashion.

Note: The discussion in the first lecture might help with a few of the problems.

1. Give the general solutions to the differential equations:
   a. $\frac{du}{dt} = 1 + 2e^{-t} - 3u$
   b. $u''(x) + u'(x) - 2u(x) = \cos(x)$
   c. $u''(x) + u(x) = 3 - e^{-x}$
2. Give the unique solution to the initial value problems:
   a. $\frac{du}{dt} = 1 + 2e^{-t} - 3u,\ u(0) = 2$
   b. $u''(x) + u'(x) - 2u(x) = \cos(x),\ u(0) = -1,\ u'(0) = 2$
   c. $u''(x) + u(x) = 3 - e^{-x},\ u(0) = 1,\ u'(0) = -1$
3. Determine whether the following boundary value problems have solutions. If so, then give the solution(s).
   a. $u''(x) - u(x) = 0,\ u(0) = 1,\ u(1) = 2$
   b. $u''(x) = \sin(2\pi x),\ u(0) = 0,\ u(1) = 0$
   c. $u''(x) - 4u(x) = 0,\ -u'(0) + u(0) = 1,\ u'(1) = 2$
   d. $u''(x) - u(x) = \int_0^1 x\,u(s)\,ds,\ -u'(0) + u(0) = 1,\ u'(1) = 2$ (think!!)
4. Suppose $f(t)$ is a continuous function, and a and u_0 are real numbers. Derive a formula for the solution to the initial value problem

$$u'(t) = a\,u(t) + f(t),\ u(0) = u_0$$

5. Suppose $u(t)$ is a nonnegative continuous function for $t \geq 0$. In addition, suppose a, b are a real numbers such that

$$u(t) \leq a + b\int_0^t u(s)\,ds, \text{ for all } t \geq 0.$$

Prove that . $u(t) \leq a\,e^{bt}$ for all $t \geq 0$.