Theorem 2.1. *There is a (randomized) algorithm that, given* $\varepsilon, \eta > 0$, *returns a real number* ζ *for which*

$$(1-\varepsilon)\zeta < \mathrm{vol}(K) < (1+\varepsilon)\zeta$$

with probability at least $1-\eta$. *The algorithm uses*

$$O\left(\frac{n^5}{\varepsilon^2}\left(\ln\frac{1}{\varepsilon}\right)^3\left(\ln\frac{1}{\eta}\right)\ln^5 n\right) = O^*(n^5)$$

oracle calls.

The proof of this Theorem is given at the end of Section 6.

As in all previous volume algorithms, the main technical tool is *sampling* from K, i.e., generating (approximately) uniformly distributed and (approximately) independent random points in K. We in fact make use of several sampling algorithms, working under slightly different assumptions. A result that has a simple statement is the following.

Theorem 2.2. *Given a convex body K satisfying* $B \subseteq K \subseteq dB$, *a positive integer N and* $\varepsilon > 0$, *we can generate a set of N random points* $\{v_1, \ldots, v_N\}$ *in K that are*

(a) *almost uniform in the sense that the distribution of each one is at most* ε *away from the uniform in total variation distance, and*
(b) *almost (pairwise) independent in the sense that for every* $1 \le i < j \le N$ *and every two measurable subsets A and B of K,*

$$\left|\mathsf{P}(v_i \in A, v_j \in B) - \mathsf{P}(v_i \in A)\mathsf{P}(v_j \in B)\right| \le \varepsilon.$$

The algorithm uses only $O^*(n^3d^2 + Nn^2d^2)$ *calls on the oracle.*

This running time represents an improvement of $O^*(n)$ over previous algorithms (see Lovász and Simonovits [23], Theorem 3.7) for this problem.

To make the sampling algorithm as efficient as possible, we have to find an affine transformation that minimizes the parameter d. Finding an affine transformation A such that

$$B \subseteq AK \subseteq d'B \tag{1}$$

for some small d' is called *rounding* or *sandwiching*. For every convex K, the sandwiching ratio $d' = n$ can be achieved (using the so called the Löwner–John ellipsoid), but it is not known how to find the corresponding A in polynomial time. For related references we again recommend Grötschel, Lovász, and Schrijver [10] and the *Handbook of Convex Geometry* [11]. For our purposes "approximate sandwiching" is sufficient, where $d'B$ is required to contain most of K but not the whole body. The theorem below will imply that that one can approximately well-round K with $d' = O(\sqrt{n}\ln(1/\varepsilon))$ using $O^*(n^5)$ oracle calls.

The approximate sandwiching will be done using an important auxiliary result, which may be of interest in its own: an algorithm to find an affine transformation