A rigorous proof mean-ergodicity implying a condition on the covariance sequence is given below.

Theorem: A WSS process $x[n]$ is mean-squared ergodic in the mean if and only if its covariance sequence satisfies

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} r_{xx}[n] = 0 \tag{1}$$

Proof: Without loss of generality we can assume that the process has zero mean. Define

$$W_N = \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]$$

We will first assume that the process is MS ergodic in the mean and show that Eq. (1) is true. From the assumption it follows that $\lim_{N\to\infty} \mathcal{E}\{W_N^2\} = 0$. To prove Eq. (1), using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\left| \frac{1}{2N+1} \sum_{n=-N}^{N} r_{xx}[n] \right|^2 &= \left| \frac{1}{2N+1} \sum_{n=-N}^{N} \mathcal{E}\{x[n]x[0]\} \right|^2 \\
&= |\mathcal{E}\{x[0]W_N\}|^2 \\
&\leq \mathcal{E}\{x^2[0]\}\mathcal{E}\{W_N^2\}
\end{aligned}$$

Hence Eq. (1) follows.
To show the converse, compute $\mathcal{E}\{W_N^2\}$ as follows:

$$\begin{aligned}
\mathcal{E}\{W_N^2\} &= \frac{1}{(2N+1)^2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} \mathcal{E}\{x[n]x[m]\} \\
&= \frac{1}{(2N+1)^2} \sum_{n=-N}^{N} \sum_{m=-N}^{N} r_{xx}[n-m] \\
&= \frac{2}{(2N+1)^2} \sum_{m=0}^{2N} \sum_{k=0}^{m} r_{xx}[k] - \frac{r_{xx}[0]}{2N+1}
\end{aligned}$$

Since we are interested in the limit as $N \to \infty$, the second term is immaterial, and we consider only the first term.

Choose $\epsilon > 0$ and let M be such that

$$\left| \sum_{k=0}^{m} r_{xx}[k] \right| < m\epsilon, \qquad \forall m > M$$

The existence of M follows from Eq. (1), which we have assumed to be true. Now, for $N > M$ we have

$$\begin{aligned}
\left| \frac{1}{(2N+1)^2} \sum_{m=0}^{2N} \sum_{k=0}^{m} r_{xx}[k] \right| &\leq \frac{1}{(2N+1)^2} \left| \sum_{m=0}^{M} \sum_{k=0}^{m} r_{xx}[k] \right| + \frac{1}{(2N+1)^2} \left| \sum_{m=M+1}^{2N} \sum_{k=0}^{m} r_{xx}[k] \right| \\
&\leq \frac{1}{(2N+1)^2} \left| \sum_{m=0}^{M} \sum_{k=0}^{m} r_{xx}[k] \right| + \frac{1}{(2N+1)^2} \sum_{m=M+1}^{2N} m\epsilon \\
&\leq \frac{1}{(2N+1)^2} \left| \sum_{m=0}^{M} \sum_{k=0}^{m} r_{xx}[k] \right| + \epsilon
\end{aligned}$$

Now let $N \to \infty$. Since M is fixed, the RHS is smaller than 2ϵ for large enough N. Hence the LHS approaches 0 as $N \to \infty$, proving that $\lim_{N\to\infty} \mathcal{E}\{W_N^2\} = 0$