# Solutions to practice final.

1 a)

Calories:
$$\underbrace{400x_1 + 300x_2 + 500x_3}_{\text{Total \# of calories}} \le 2000.$$

Calories from fat: 80 for Special L
90 for " M
200 for " N

so
$$80x_1 + 90x_2 + 200x_3 \le .35(400x_1 + 300x_2 + 500x_3)$$

(can simplify to
$$-60x_1 - 15x_2 + 25x_3 \le 0.)$$

b) Add dummy machine. Assigning job to dummy means not processed:

| Jobs \ Machines | 1 | 2 | 3 | 4 = dummy | Supplies |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 5 | 1 |
| 2 | 2 | 4 | 6 | 10 | 1 |
| 3 | 3 | 6 | 9 | 15 | 1 |
| 4 | 4 | 8 | 12 | 20 | 1 |
| Demands | 1 | 1 | 1 | 1 | |

2

a) $$\begin{aligned} \min \quad & 3y_1 + 6y_2 + 4y_3 \\ & y_1 + 3y_2 + 2y_3 \geq 30 \\ & -y_1 - 2y_2 \geq +1 \quad \Longleftarrow \\ & 4y_1 + y_2 + y_3 \geq 20 \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

Consider marked constr. For nonneg $y_i$'s, LHS $\leq 0$, but RHS $> 0$, so can't be satisfied. Infeas dual means primal infeas or unbded.

b) Find $B^{-1}b = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{array}{l} \leftarrow x_4 \\ \leftarrow x_5 \\ \leftarrow x_1 \end{array}$

So bf sol'n is $x = (2, 0, 0, 1, \underset{\substack{\uparrow \\ \text{basic}}}{0}, 0)^T$.

c) Dual sol'n is $y = c_B B^{-1} =$

$$[0, 0, 30] \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1/2 \end{pmatrix} = (0, 0, 15).$$

d) Coeffs of $x_1, x_2, x_3$ (know that of $x_1 = 0$, so just a check) from $yA_j - c_j$:

$$j=1: \quad (0,0,15)\begin{pmatrix}1\\3\\2\end{pmatrix} - 30 = 30-30 = 0 \checkmark$$

$$j=2: \quad (0,0,15)\begin{pmatrix}-1\\-2\\0\end{pmatrix} - 1 = 0-1 = -1.$$

$$j=3: \quad (0,0,15)\begin{pmatrix}2\\0\\1\end{pmatrix} - 20 = 15-20 = -5.$$

So usual rule: most neg row 0 coeff, so $x_3$ entering bv.

Compute pivot col: $B^{-1}A_3 = \begin{pmatrix}1 & 0 & -1/2\\0 & 1 & -3/2\\0 & 0 & 1/2\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}3/2\\-3/2\\1/2\end{pmatrix}$;

with RHS $\begin{pmatrix}1\\0\\2\end{pmatrix}$ min ratio test gives min $\frac{1}{3/2}$ in row 1, so first basic var or $x_4$ leaving bv.

(Note: if make pivot, next bf soln nondeg.)

e) Bland's rule: first neg row 0 coeff, so $x_2$ enters.

Compute pivot col. $B^{-1}A_2 = \begin{pmatrix}1 & 0 & -1/2\\0 & 1 & -3/2\\0 & 0 & 1/2\end{pmatrix}\begin{pmatrix}-1\\-2\\0\end{pmatrix} = \begin{pmatrix}-1\\-2\\0\end{pmatrix}$

All $\leq 0$, so no leaving variable. Indication of unboundedness. (If want obj fn value $> 1000$, have current value of 60 ($c_B B^{-1} b$), so want $> 940$

more; since $-1$ row 0 coeff indicates improvement per unit of entering bv, want $x_2 > 940$. Say we set $x_2 = 1000$. Then from entries in pivot col, $x_4$ becomes $1 + 1 \cdot 1000 = 1001$, $x_5$ becomes $0 + 2 \cdot 1000 = 2000$, and $x_1$ becomes $2 + 0 \cdot 1000 = 2$. Get (not basic) feasible sol'n $x = (2, 1000, 0, 1001, 2000, 0)^T$, with obj fn value $60 + 1 \cdot 1000 = 1060$.)

(f) $x_3$ enters instead of $x_4$, so new basic vars $x_3, x_5, x_1$ in that order. Get elem matrix

$$E = \begin{pmatrix} 2/7 & 0 & 0 \\ +1/7 & 1 & 0 \\ -1/7 & 0 & 1 \end{pmatrix}$$

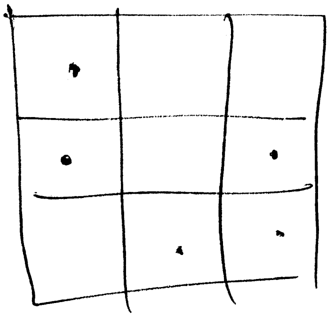
from pivot col, and

$$(B^{-1})_{new} = EB^{-1} = \begin{pmatrix} 2/7 & 0 & 0 \\ 1/7 & 1 & 0 \\ -1/7 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1/2 \end{pmatrix} =$$

$$\cancel{\begin{pmatrix} 2/3 & 0 & -1/3 \\ 1 & 1 & -2 \\ -1/3 & 0 & 2/3 \end{pmatrix}} \begin{pmatrix} 2/7 & 0 & -1/7 \\ 1/7 & 1 & -11/7 \\ -1/7 & 0 & 4/7 \end{pmatrix}.$$

3.

a)



From 1st supply Constr., $x_{11} = 20$

—— demand

——, $x_{21} = 30 - x_{11} = 10$

From 2nd supply

——, $x_{23} = 10 - x_{21} = 0$

From 3rd demand

——, $x_{33} = 30 - x_{23} = 30$,

From 3rd supply constraint,

$x_{32} = 40 - x_{33} = 10$

---

b) $u_3 = 0$. From $x_{32}$ basic, $u_3 + v_2 = c_{32} = 4$, $v_2 = 4$

—— $x_{33}$ basic, $v_3 = c_{33} - u_3 = 3$

—— $x_{23}$ basic, $u_2 = c_{23} - v_3 = 2$

—— $x_{21}$ ——, $v_1 = c_{21} - u_2 = 5$

—— $x_{11}$ ——, $u_1 = c_{11} - v_1 = -1$.

c)

| | | | | $s_i$'s | $u_i$'s |
|---|---|---|---|---|---|
| | 4) $-\theta$ 20 | 1) $+\theta$ $-2$ | 6) $+4$ | 20 | $-1$ |
| | 7) $+\theta$ 10 | 6) 0 | 5) $-\theta$ 0 | 10 | 2 |
| | 6) $+1$ | 4) $-\theta$ 10 | 3) $+\theta$ 30 | 40 | 0 |
| $d_j$'s | 30 | 10 | 30 | | |
| $v_j$'s | 5 | 4 | 3 | | |

○ basic flows

$c_{ij} - u_i - v_j$

So $x_{12}$ entering bv.

Flows change as indicated. So max $\theta = 0$, & $x_{12}$ becomes basic instead of $x_{23}$, but at level 0. New bf sol'n

| | | |
|---|---|---|
| 20 | 0 | |
| 10 | | |
| | 10 | 30 |

Since entering variable entered at level 0, cost did not change

4.

a) $\max\ 5x_1 + 2x_2 + \mu \ln(3-2x_1) + \mu \ln(2-x_2) + \mu \ln(7-3x_1-2x_2) + \mu \ln x_1 + \mu \ln x_2.$

b) Gradient is

$$\begin{pmatrix} 5 - \dfrac{2\mu}{3-2x_1} - \dfrac{3\mu}{7-3x_1-2x_2} + \dfrac{\mu}{x_1} \\[2ex] 2 - \dfrac{\mu}{2-x_2} - \dfrac{2\mu}{7-3x_1-2x_2} + \dfrac{\mu}{x_2} \end{pmatrix}$$

With $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, gradient is

$$\begin{pmatrix} 5-2\mu - \frac{3\mu}{2} + \mu \\ 2-\mu - \frac{2\mu}{2} + \mu \end{pmatrix} = \begin{pmatrix} 5 - \frac{5}{2}\mu \\ 2-\mu \end{pmatrix}.$$

This is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $\mu = 2$. So have stationary pt at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ when $\mu = 2$. Since composite fn is (strictly) concave, have global maximizer.

c) From (b), have $x(\mu) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $\mu = 2$.

Then $s(\mu)$ is vector of slacks, $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Since $y_i s_i = \mu$ & $x_j t_j = \mu$ (all $i, j$) when on central path, get

$$y(\mu) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad t(\mu) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Can check this is feasible for dual problem.

d) $c^T x(\mu) = (5, 2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 7$.

$b^T y(\mu) = (3, 2, 7)\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 17$, so

duality gap is $b^T y(\mu) - c^T x(\mu) = 17 - 7 = 10$.

For points on central path, gap is $(m+n)\mu$.

Here $(3+2) \cdot 2 = 10$.

5.

a) Need to solve an $m \times m$ system of equations, $M \cdot \Delta y = v$, with $M$ a nonsingular matrix (in fact symmetric, positive definite). Seems like systems $yB = c_B$ and $Bv = A_j$ to get dual sol'n + pivot column in simplex method, but in that case $B$ changes by just one column each it'n, so can easily update $B^{-1}$. Here $M$ changes completely from one iteration to next, so need to solve from scratch.

b) $y_i$ is a tentative price associated to the ith resource. (Can interpret revised simplex method as using these prices to determine entering variable. If $y_i < 0$, want to use less resource $i$, so can increase slack var. for res $i$; if $yA_j < c_j$, then product $j$ generates more profit than it costs in resources, so increase $x_j$.)

c) Obj fn value decreases or stays the same at each it'n of the dual simplex method. It <u>can</u> stay the same if there is a zero row 0 coeff for a nonbasic var (dual degeneracy), + <u>will</u> stay the same if there is a neg coeff in the pivot row corres to such a 0 coeff in row 0.