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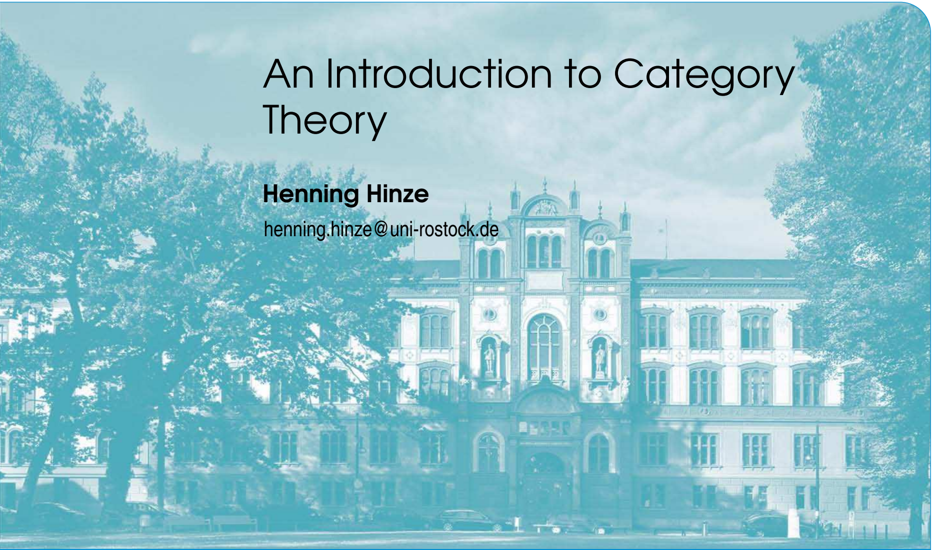

An Introduction to Category Theory

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Categories

Definition (Category)

A category $\mathbf{A}$ consists of

- a collection $ob(\mathbf{A})$ of **objects**
- for each $a, b \in ob(\mathbf{A})$ a collection $Hom_{\mathbf{A}}(a, b)$ of morphisms
- for each $a \in ob(\mathbf{A})$ a morphism $1_a \in Hom_{\mathbf{A}}(a, a)$ called the **identity** on a ($id_a = 1_a$)
- for each $a, b, c \in ob(\mathbf{A})$ a map

$$\circ : Hom_{\mathbf{A}}(a, b) \times Hom_{\mathbf{A}}(b, c) \to Hom_{\mathbf{A}}(a, c)$$
$$(f, g) \mapsto g \circ f$$

Categories

Definition (Category)

- **associativity:** for each $f \in Hom_{\mathbf{A}}(a,b), g \in Hom_{\mathbf{A}}(b,c), h \in Hom_{\mathbf{A}}(c,d)$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

- **identity:** for each $f \in Hom_{\mathbf{A}}(a,b)$ we have

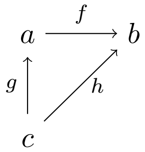
$$f \circ 1_a = f = 1_b \circ f$$

Examples

- $\mathbf{Set}$
- $\mathbf{Grp}$
- $\mathbf{Vect}_k$
- $\mathbf{C}$ with
  - $ob(\mathbf{C}) = \{X\}$
  - $Hom_{\mathbf{C}}(X, X) := \{f : X \to X \mid \text{ f bijective }\}$
  - $(Hom_{\mathbf{C}}(X, X); \circ) \in ob(\mathbf{Grp})$

Notation of categories

Categories are noted as diagrams

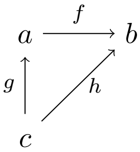


which show the objects and the morphisms between them.
For an morphism $f : a \to b$ we call

- $a = dom(f)$ the **domain** of f and
- $b = cod(f)$ the **codomain** of f.

Notation of categories

Categories are noted as diagrams



which show the objects and the morphisms between them.
We say the diagram **commutes** if

$$h = f \circ g.$$

Notation of categories

Categories are noted as diagrams

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ h\uparrow & & \uparrow j \\ c & \xrightarrow{g} & d \end{array}$$

which show the objects and the morphisms between them.
We say the diagram **commutes** if

$$f \circ h = j \circ g.$$

Properties of Categories

Definition

Let $\mathbf{A}$ be a category. Then we call $\mathbf{A}$

- **small** if $ob(\mathbf{A})$ and $Hom_{\mathbf{A}}(a, b)$, for all $a, b \in ob(\mathbf{A})$, are actually sets
- **locally small** if $Hom_{\mathbf{A}}(a, b)$ is a set for each $a, b \in ob(\mathbf{A})$
- **large** otherwise

Recap group isomorphisms

Definition (Group homomorphisms)

Let $(G;\star)$ and $(H;\Box)$ be groups. A group homomorphism is a map

$$\varphi : G \to H$$

s.t. for all $g_1, g_2 \in G$

$$\varphi(g_1 \star g_2) = \varphi(g_1) \Box \varphi(g_2).$$

Recap group isomorphism

Definition (Group isomorphism)

Let $(G;\star)$ and $(H;\Box)$ be groups. A group isomorphism is a map

$$\varphi : G \to H$$

s.t.

- φ is a group homomorphism and
- there exists a map $\varphi^{-1} : H \to G$ with

$$\varphi \circ \varphi^{-1} = id_H \text{ and } \varphi^{-1} \circ \varphi = id_G$$

Recap group homomorphisms

Example

Consider the groups

$\star$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

and

$\square$	α	β	γ	δ
α	α	β	γ	δ
β	β	δ	α	γ
γ	γ	α	δ	β
δ	δ	γ	β	α

with

$$\varphi(0) = \alpha, \quad \varphi(1) = \beta, \quad \varphi(2) = \delta, \quad \varphi(3) = \gamma$$

the groups are isomorphic and φ is the isomorphism.

Product categories

Definition (Product category)

Let $\mathbf{A}$ and $\mathbf{X}$ be categories. Then the **product category** $\mathbf{A} \times \mathbf{X}$ of $\mathbf{A}$ and $\mathbf{X}$ looks like this:

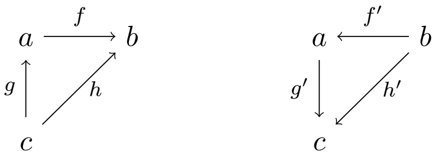
- $ob(\mathbf{A} \times_c \mathbf{X}) := ob(\mathbf{A}) \times ob(\mathbf{X})$
- $Hom_{\mathbf{A}\times_c\mathbf{X}}((a,x),(a',x')) := Hom_{\mathbf{A}}(a,a') \times Hom_{\mathbf{X}}(x,x')$ for $a, a' \in ob(\mathbf{A}), x, x' \in ob(\mathbf{X})$

Duality

Definition (Dual Category)

For each category $\mathbf{A}$ there is a **dual category** $\mathbf{A}^{op}$ s.t.

- $ob(\mathbf{A}^{op}) = ob(\mathbf{A})$
- for $f \in Hom_{\mathbf{A}}(a, b)$ there is $f' \in Hom_{\mathbf{A}^{op}}(b, a)$



Products

Definition (Product)

Let $\mathbf{C}$ be a category and $X, Y \in ob(\mathbf{C})$. A **product of X and Y** is

$$(P, p_1 : P \to X, p_2 : P \to Y)$$

with $P \in ob(\mathbf{C})$ s.t. for all

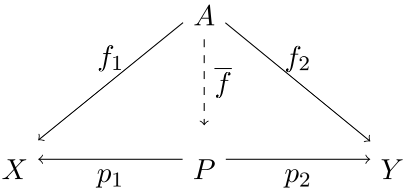
$$(A, f_1 : A \to X, f_2 : A \to Y)$$

with $A \in ob(\mathbf{C})$ there is a unique $\overline{f} : A \to P$ with

$$f_1 = p_1 \circ \overline{f}, \qquad f_2 = p_2 \circ \overline{f}.$$

Products

Definition



commutes.

Coproducts

Definition (Coproduct)

Let $\mathbf{C}$ be a category and $X, Y \in ob(\mathbf{C})$. A **coproduct of X and Y** is

$$(P, P \leftarrow X : i_1, P \leftarrow Y : i_2)$$

with $P \in ob(\mathbf{C})$ s.t. for all

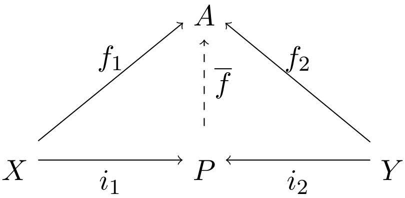
$$(A, A \leftarrow X : f_1, A \leftarrow Y : f_2)$$

with $A \in ob(\mathbf{C})$ there is a unique $A \leftarrow P : \overline{f}$ with

$$f_1 = \overline{f} \circ i_1, \qquad f_2 = \overline{f} \circ i_2.$$

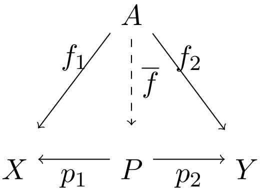
Coproducts

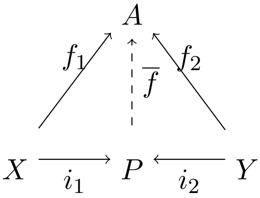
Definition



commutes.

Product/Coproduct





Functors

Definition (Functor)

Let $\mathbf{A}$ and $\mathbf{X}$ be categories. A Functor $T : \mathbf{A} \to \mathbf{X}$ is a map s.t.

- for all $a \in ob(\mathbf{A})$ we have $T(a) \in ob(\mathbf{X})$
- for all $a, a' \in ob(\mathbf{A})$ and $f \in Hom_{\mathbf{A}}(a, a')$ we have $T(f) \in Hom_{\mathbf{X}}(T(a), T(a'))$

which satisfies the following axioms:

- for each $a \in ob(\mathbf{A})$ we have $T(1_a) = 1_{T(a)}$
- for $f \in Hom_{\mathbf{A}}(a, a'), f' \in Hom_{\mathbf{A}}(a', a'')$ we have

$$T(f' \circ f) = T(f') \circ T(f)$$

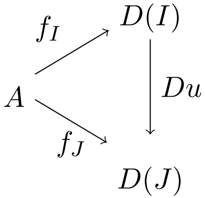
Examples for Functors

- Identity functor: $id_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$
- Forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ 'forgets group structure': A Group $(G; \star)$ is mapped to the underlying set G
- Free functor $F : \mathbf{Set} \to \mathbf{Grp}$ maps a set to the smallest group containing that set
- Bifunctors: $B : \mathbf{C}^{op} \times_c \mathbf{C} \to \mathbf{D}$
  - Hom-bifunctor: $Hom : \mathbf{C}^{op} \times_c \mathbf{C} \to \mathbf{Set}$
- Covariant Hom-functor: $Hom_{\mathbf{C}}(c, -) : \mathbf{C} \to \mathbf{Set}$
- Contravariant Hom-functor: $Hom_{\mathbf{C}}(-, c) : \mathbf{C}^{op} \to \mathbf{Set}$

Cones

Definition (Cone)

Let $\mathbf{A}$ and $\mathbf{I}$ be categories and $D : \mathbf{I} \to \mathbf{A}$ be a functor. A **cone on** D is an object $A \in \mathbf{A}$ and a family of morphisms $(A \xrightarrow{f_I} D(I))_{I \in ob(\mathbf{I})}$ s.t. for all $u : I \to J$ in $\mathbf{I}$



commutes.

Natural Transformations

Definition (Component)

Let $\mathbf{A}$ and $\mathbf{X}$ be categories, $F, G : \mathbf{A} \to \mathbf{X}$ be functors and $f \in Hom_{\mathbf{A}}(c, c')$ for $c, c' \in ob(\mathbf{A})$. A **component** is a morphism $\alpha_c \in Hom_{\mathbf{X}}(Fc, Gc)$ s.t.

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow[\alpha_{c'}]{} & Gc' \end{array}$$

commutes.

Natural Transformations

Definition (Natural Transformation)

Let $\mathbf{A}$ and $\mathbf{X}$ be categories and $F, G : \mathbf{A} \to \mathbf{X}$ be functors. A **natural transformation** $\alpha : F \Rightarrow G$ is a map assigning to every $c \in ob(\mathbf{A})$ its component $\alpha_c \in Hom_{\mathbf{X}}(Fc, Gc)$.

Adjoints

Definition (Adjoint)

Let $\mathbf{A}$ and $\mathbf{X}$ be categories and $F : \mathbf{A} \rightleftarrows \mathbf{X} : G$ be functors. Then F is **left adjoint** to G and G is **right adjoint** to F if

$$\varphi_{x,a} : Hom_{\mathbf{X}}(Fa, x) \cong Hom_{\mathbf{A}}(a, Gx)$$

and for

$$a, a' \in ob(\mathbf{A}) \qquad k \in Hom_{\mathbf{A}}(a', a)$$
$$x, x' \in ob(\mathbf{X}) \qquad h \in Hom_{\mathbf{X}}(x, x')$$

...

Adjoints

Definition (Adjoint)

...

$$\begin{array}{ccc}
Hom_{\mathbf{X}}(Fa, x) & \xrightarrow{\varphi_{x,a}} & Hom_{\mathbf{A}}(a, Gx) \\
{\scriptstyle Hom_{\mathbf{X}}(Fa,h)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathbf{A}}(a,Gh)} \\
Hom_{\mathbf{X}}(Fa, x') & \xrightarrow[\varphi_{x',a}]{} & Hom_{\mathbf{A}}(a, Gx')
\end{array}$$

and

$$\begin{array}{ccc}
Hom_{\mathbf{X}}(Fa, x) & \xrightarrow{\varphi_{x,a}} & Hom_{\mathbf{A}}(a, Gx) \\
{\scriptstyle Hom_{\mathbf{X}}(Fk,x)}\downarrow & & \downarrow{\scriptstyle Hom_{\mathbf{A}}(k,Gx)} \\
Hom_{\mathbf{X}}(Fa', x) & \xrightarrow{\varphi_{x,a'}} & Hom_{\mathbf{A}}(a', Gx)
\end{array}$$

commute.

Example

$$V : \mathbf{Set} \rightleftarrows \mathbf{Vect_k} : U$$

take $X \in ob(\mathbf{Set}), W \in ob(\mathbf{Vect_k})$

$$\varphi_{X,W} : Hom_{\mathbf{Vect_k}}(V(X), W) \cong Hom_{\mathbf{Set}}(X, U(W))$$

(Co)units

Definition ((Co)unit)

Every adjunction $F : \mathbf{A} \rightleftarrows \mathbf{X} : G$ yields two natural transformations:

- the **unit** natural transformation

$$\eta : 1_{\mathbf{A}} \Rightarrow G \circ F$$
$$\eta_A : A \to GFA$$

- the **counit** natural transformation

$$\epsilon : F \circ G \Rightarrow 1_{\mathbf{X}}$$
$$\epsilon_X : FGX \to X$$

Literature

- (*) Basic Category Theory - Tom Leinster
- (*) Categories and Computer Science - RFC Walters
- (***) Categories for the working mathematician - Saunders Mac Lane
- (***) ncatlab.org
- (**) en.wikipedia.org