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Research Article

Solution and Stability of a Mixed Type Additive, Quadratic, and Cubic Functional Equation

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We obtain the general solution and the generalized Hyers-Ulam-Rassias stability of the mixed type additive, quadratic, and cubic functional equation $f(x+2y)-f(x-2y)=2(f(x+y)-f(x-y))+2f(3y)-6f(2y)+6f(y)$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group, and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In other words, under what condition does there exist a homomorphism near an approximate homomorphism?

In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta, \tag{1.1}$$

for all $x,y\in E$ and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\|f(x)-T(x)\|\le\delta, \tag{1.2}$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in t for each fixed $x \in E$, then T is linear (see also [3]). In 1950, Aoki [4] generalized Hyers' theorem for approximately additive mappings. In 1978, Th. M. Rassias [5] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [2–24]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is related to symmetric biadditive function. In the real case it has $f(x) = x^2$ among its solutions. Thus, it has been called quadratic functional equation, and each of its solutions is said to be a quadratic function. Hyers-Ulam-Rassias stability for the quadratic functional equation (1.3) was proved by Skof for functions $f : A \to B$, where A is normed space and B Banach space (see [25–28]).

The following cubic functional equation was introduced by the third author of this paper, J. M. Rassias [29, 30] (in 2000-2001):

$$f(x+2y) + 3f(x) = 3f(x+y) + f(x-y) + 6f(y). \tag{1.4}$$

Jun and Kim [13] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.5}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5).

The function $f(x) = x^3$ satisfies the functional equation (1.5), which explains why it is called cubic functional equation.

Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables (see also [31–33]).

We deal with the following functional equation deriving from additive, cubic and quadratic functions:

$$f(x+2y) - f(x-2y) = 2(f(x+y) - f(x-y)) + 2f(3y) - 6f(2y) + 6f(y). \tag{1.6}$$

It is easy to see that the function $f(x) = ax^3 + bx^2 + cx$ is a solution of the functional equation (1.6). In the present paper we investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.6).

2. General Solution

In this section we establish the general solution of functional equation (1.6).

Theorem 2.1. *Let X,Y be vector spaces, and let $f : X \to Y$ be a function. Then f satisfies* (1.6) *if and only if there exists a unique additive function $A : X \to Y$, a unique symmetric and biadditive function $Q : X \times X \to Y$, and a unique symmetric and 3-additive function $C : X \times X \times X \to Y$ such that $f(x) = A(x) + Q(x,x) + C(x,x,x)$ for all $x \in X$.*

Proof. Suppose that $f(x) = A(x) + Q(x,x) + C(x,x,x)$ for all $x \in X$, where $A : X \to Y$ is additive, $Q : X \times X \to Y$ is symmetric and biadditive, and $C : X \times X \times X \to Y$ is symmetric and 3-additive. Then it is easy to see that f satisfies (1.6). For the converse let f satisfy (1.6). We decompose f into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \qquad f_o(x) = \frac{1}{2}(f(x) - f(-x)), \tag{2.1}$$

for all $x \in X$. By (1.6), we have

$$\begin{aligned}
&f_e(x+2y) - f_e(x-2y) \\
&\quad= \frac{1}{2}[f(x+2y) + f(-x-2y) - f(x-2y) - f(-x+2y)] \\
&\quad= \frac{1}{2}[f(x+2y) - f(x-2y)] + \frac{1}{2}[f((-x)+(-2y)) - f((-x)-(-2y))] \\
&\quad= \frac{1}{2}[2f(x+y) - 2f(x-y) + 2f(3y) - 6f(2y) + 6f(y)] \\
&\qquad+ \frac{1}{2}[2f(-x-y) - 2f(-x+y) + 2f(-3y) - 6f(-2y) + 6f(-y)] \\
&\quad= 2\left[\frac{1}{2}(f(x+y) + f(-x-y))\right] - 2\left[\frac{1}{2}(f(x-y) + f(-x+y))\right] \\
&\qquad+ 2\left[\frac{1}{2}(f(3y) + f(-3y))\right] - 6\left[\frac{1}{2}(f(2y) + f(-2y))\right] + 6\left[\frac{1}{2}(f(y) + f(-y))\right] \\
&\quad= 2(f_e(x+y) - f_e(x-y)) + 2f_e(3y) - 6f_e(2y) + 6f_e(y),
\end{aligned} \tag{2.2}$$

for all $x, y \in X$. This means that f_e satisfies (1.6), that is,

$$f_e(x+2y) - f_e(x-2y) = 2(f_e(x+y) - f_e(x-y)) + 2f_e(3y) - 6f_e(2y) + 6f_e(y). \tag{2.3}$$

Now putting $x = y = 0$ in (2.3), we get $f_e(0) = 0$. Setting $x = 0$ in (2.3), by evenness of f_e we obtain

$$3f_e(2y) = f_e(3y) + 3f_e(y). \tag{2.4}$$

Replacing x by y in (2.3), we obtain

$$4f_e(2y) = f_e(3y) + 7f_e(y). \tag{2.5}$$

Comparing (2.4) with (2.5), we get

$$f_e(3y) = 9f_e(y). \tag{2.6}$$

By utilizing (2.5) with (2.6), we obtain

$$f_e(2y) = 4f_e(y). \tag{2.7}$$

Hence, according to (2.6) and (2.7), (2.3) can be written as

$$f_e(x+2y) - f_e(x-2y) = 2f_e(x+y) - 2f_e(x-y). \tag{2.8}$$

With the substitution $x := x+y,\ y := x-y$ in (2.8), we have

$$f_e(3x-y) - f_e(x-3y) = 8f_e(x) - 8f_e(y). \tag{2.9}$$

Replacing y by $-y$ in above relation, we obtain

$$f_e(3x+y) - f_e(x+3y) = 8f_e(x) - 8f_e(y). \tag{2.10}$$

Setting $x+y$ instead of x in (2.8), we get

$$f_e(x+3y) - f_e(x-y) = 2f_e(x+2y) - 2f_e(x). \tag{2.11}$$

Interchanging x and y in (2.11), we get

$$f_e(3x+y) - f_e(x-y) = 2f_e(2x+y) - 2f_e(y). \tag{2.12}$$

If we subtract (2.12) from (2.11) and use (2.10), we obtain

$$f_e(x+2y) - f_e(2x+y) = 3f_e(y) - 3f_e(x), \tag{2.13}$$

which, by putting $y := 2y$ and using (2.7), leads to

$$f_e(x+4y) - 4f_e(x+y) = 12f_e(y) - 3f_e(x). \tag{2.14}$$

Let us interchange x and y in (2.14). Then we see that

$$f_e(4x+y) - 4f_e(x+y) = 12f_e(x) - 3f_e(y), \tag{2.15}$$

and by adding (2.14) and (2.15), we arrive at

$$f_e(x+4y) + f_e(4x+y) = 8f_e(x+y) + 9f_e(x) + 9f_e(y). \tag{2.16}$$

Replacing y by $x + y$ in (2.8), we obtain

$$f_e(3x + 2y) - f_e(x + 2y) = 2f_e(2x + y) - 2f_e(y). \tag{2.17}$$

Let us Interchange x and y in (2.17). Then we see that

$$f_e(2x + 3y) - f_e(2x + y) = 2f_e(x + 2y) - 2f_e(x). \tag{2.18}$$

Thus by adding (2.17) and (2.18), we have

$$f_e(2x + 3y) + f_e(3x + 2y) = 3f_e(x + 2y) + 3f_e(2x + y) - 2f_e(x) - 2f_e(y). \tag{2.19}$$

Replacing x by $2x$ in (2.11) and using (2.7) we have

$$f_e(2x + 3y) - f_e(2x - y) = 8f_e(x + y) - 8f_e(x), \tag{2.20}$$

and interchanging x and y in (2.20) yields

$$f_e(3x + 2y) - f_e(x - 2y) = 8f_e(x + y) - 8f_e(y). \tag{2.21}$$

If we add (2.20) to (2.21), we have

$$f_e(2x + 3y) + f_e(3x + 2y) = f_e(2x - y) + f_e(x - 2y) + 16f_e(x + y) - 8f_e(x) - 8f_e(y). \tag{2.22}$$

Interchanging x and y in (2.8), we get

$$f_e(2x + y) - f_e(2x - y) = 2f_e(x + y) - 2f_e(x - y), \tag{2.23}$$

and by adding the last equation and (2.8) with (2.19), we get

$$\begin{aligned} &f_e(2x + 3y) + f_e(3x + 2y) - f_e(2x - y) - f_e(x - 2y) \\ &\quad = 2f_e(x + 2y) + 2f_e(2x + y) + 4f_e(x + y) - 4f_e(x - y) - 2f_e(x) - 2f_e(y). \end{aligned} \tag{2.24}$$

Now according to (2.22) and (2.24), it follows that

$$f_e(x + 2y) + f_e(2x + y) = 6f_e(x + y) + 2f_e(x - y) - 3f_e(x) - 3f_e(y). \tag{2.25}$$

From the substitution $y = -y$ in (2.25) it follows that

$$f_e(x - 2y) + f_e(2x - y) = 6f_e(x - y) + 2f_e(x + y) - 3f_e(x) - 3f_e(y). \tag{2.26}$$

Replacing y by $2y$ in (2.25) we have

$$f_e(x+4y)+4f_e(x+y)=6f_e(x+2y)+2f_e(x-2y)-3f_e(x)-12f_e(y), \tag{2.27}$$

and interchanging x and y yields

$$f_e(4x+y)+4f_e(x+y)=6f_e(2x+y)+2f_e(2x-y)-12f_e(x)-3f_e(y). \tag{2.28}$$

By adding (2.27) and (2.28) and then using (2.25) and (2.26), we lead to

$$f_e(x+4y)+f_e(4x+y)=32f_e(x+y)+24f_e(x-y)-39f_e(x)-39f_e(y). \tag{2.29}$$

If we compare (2.16) and (2.29), we conclude that

$$f_e(x+y)+f_e(x-y)=2f_e(x)+2f_e(y). \tag{2.30}$$

This means that f_e is quadratic. Thus there exists a unique quadratic function $Q : X\times X \to Y$ such that $f_e(x)=Q(x,x)$, for all $x\in$ X. On the other hand we can show that f_o satisfies (1.6), that is,

$$f_o(x+2y)-f_o(x-2y)=2(f_o(x+y)-f_o(x-y))+2f_o(3y)-6f_o(2y)+6f_o(y). \tag{2.31}$$

Now we show that the mapping $g : X \to Y$ defined by $g(x) := f_o(2x)-8f_o(x)$ is additive and the mapping $h : X \to Y$ defined by $h(x) := f_o(2x)-2f_o(x)$ is cubic. Putting $x=0$ in (2.31), then by oddness of f_o, we have

$$4f_o(2y)=5f_o(y)+f_o(3y). \tag{2.32}$$

Hence (2.31) can be written as

$$f_o(x+2y)-f_o(x-2y)=2f_o(x+y)-2f_o(x-y)+2f_o(2y)-4f_o(y). \tag{2.33}$$

From the substitution $y := -y$ in (2.33) it follows that

$$f_o(x-2y)-f_o(x+2y)=2f_o(x-y)-2f_o(x+y)-2f_o(2y)+4f_o(y). \tag{2.34}$$

Interchange x and y in (2.33), and it follows that

$$f_o(2x+y)+f_o(2x-y)=2f_o(x+y)+2f_o(x-y)+2f_o(2x)-4f_o(x). \tag{2.35}$$

With the substitutions $x := x-y$ and $y := x+y$ in (2.35), we have

$$f_o(3x-y)+f_o(x-3y)=2f_o(2x-2y)-4f_o(x-y)+2f_o(2x)-2f_o(2y). \tag{2.36}$$

Replace x by $x-y$ in (2.34). Then we have

$$f_o(x-3y) - f_o(x+y) = 2f_o(x-2y) - 2f_o(x) - 2f_o(2y) + 4f_o(y). \tag{2.37}$$

Replacing y by $-y$ in (2.37) gives

$$f_o(x+3y) - f_o(x-y) = 2f_o(x+2y) - 2f_o(x) + 2f_o(2y) - 4f_o(y). \tag{2.38}$$

Interchanging x and y in (2.38), we get

$$f_o(3x+y) + f_o(x-y) = 2f_o(2x+y) - 2f_o(y) + 2f_o(2x) - 4f_o(x). \tag{2.39}$$

If we add (2.38) to (2.39), we have

$$\begin{aligned} &f_o(x+3y) + f_o(3x+y) \\ &\quad = 2f_o(x+2y) + 2f_o(2x+y) + 2f_o(2x) + 2f_o(2y) - 6f_o(x) - 6f_o(y). \end{aligned} \tag{2.40}$$

Replacing y by $-y$ in (2.36) gives

$$f_o(x+3y) + f_o(3x+y) = 2f_o(2x+2y) - 4f_o(x+y) + 2f_o(2x) + 2f_o(2y). \tag{2.41}$$

By comparing (2.40) with (2.41), we arrive at

$$f_o(x+2y) + f_o(2x+y) = f_o(2x+2y) - 2f_o(x+y) + 3f_o(x) + 3f_o(y). \tag{2.42}$$

Replacing y by $-y$ in (2.42) gives

$$f_o(x-2y) + f_o(2x-y) = f_o(2x-2y) - 2f_o(x-y) + 3f_o(x) - 3f_o(y). \tag{2.43}$$

With the substitution $y := x+y$ in (2.43), we have

$$f_o(x-y) - f_o(x+2y) = -f_o(2y) - 3f_o(x+y) + 3f_o(x) + 2f_o(y), \tag{2.44}$$

and replacing $-y$ by y gives

$$f_o(x+y) - f_o(x-2y) = f_o(2y) - 3f_o(x-y) + 3f_o(x) - 2f_o(y). \tag{2.45}$$

Let us interchange x and y in (2.45). Then we see that

$$f_o(x+y) + f_o(2x-y) = f_o(2x) + 3f_o(x-y) - 2f_o(x) + 3f_o(y). \tag{2.46}$$

If we add (2.45) to (2.46), we have

$$f_o(2x-y)-f_o(x-2y)=f_o(2x)-2f_o(x+y)+f_o(x)+f_o(2y)+f_o(y). \tag{2.47}$$

Adding (2.42) to (2.47) and using (2.33) and (2.35), we obtain

$$f_o(2(x+y))-8f_o(x+y)=\left[f_o(2x)-8f_o(x)\right]+\left[f_o(2y)-8f_o(y)\right], \tag{2.48}$$

for all $x, y \in X$. The last equality means that

$$g(x+y)=g(x)+g(y), \tag{2.49}$$

for all $x, y \in X$. Therefore the mapping $g : X \to Y$ is additive. With the substitutions $x := 2x$ and $y := 2y$ in (2.35), we have

$$f_o(4x+2y)+f_o(4x-2y)=2f_o(2x+2y)+2f_o(2x-2y)+2f_o(4x)-4f_o(2x). \tag{2.50}$$

Let $g : X \to Y$ be the additive mapping defined above. It is easy to show that f_o is cubic-additive function. Then there exists a unique function $C : X \times X \times X \to Y$ and a unique additive function $A : X \to Y$ such that $f_o(x) = C(x, x, x) + A(x)$, for all $x \in X$, and C is symmetric and 3-additive. Thus for all $x \in X$, we have

$$f(x)=f_e(x)+f_o(x)=Q(x,x)+C(x,x,x)+A(x). \tag{2.51}$$

This completes the proof of theorem. □

The following corollary is an alternative result of Theorem 2.1.

Corollary 2.2. *Let X,Y be vector spaces, and let $f : X \to Y$ be a function satisfying* (1.6)*. Then the following assertions hold.*

(a) *If f is even function, then f is quadratic.*

(b) *If f is odd function, then f is cubic-additive.*

3. Stability

We now investigate the generalized Hyers-Ulam-Rassias stability problem for functional equation (1.6). From now on, let X be a real vector space, and let Y be a Banach space. Now before taking up the main subject, given $f : X \to Y$, we define the difference operator $D_f : X \times X \to Y$ by

$$D_f(x,y)=f(x+2y)-f(x-2y)-2\left[f(x+y)-f(x-y)\right]-2f(3y)+6f(2y)-6f(y), \tag{3.1}$$

for all $x, y \in X$. We consider the following functional inequality:

$$\|D_f(x,y)\| \le \phi(x,y), \tag{3.2}$$

for an upper bound $\phi : X \times X \to [0,\infty)$.

Theorem 3.1. *Let $s \in \{1,-1\}$ be fixed. Suppose that an even mapping $f : X \to Y$ satisfies $f(0) = 0$, and*

$$\|D_f(x,y)\| \le \phi(x,y), \tag{3.3}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0,\infty)$ is a mapping such that

$$\sum_{i=0}^{\infty} 4^{si}\left[\phi\left(2^{-si}x, 2^{-si}x\right) + \frac{1}{2}\phi\left(0, 2^{-si}x\right)\right] < \infty \tag{3.4}$$

and that

$$\lim_{n} 4^{sn}\phi(2^{-sn}x, 2^{-sn}y) = 0, \tag{3.5}$$

for all $x, y \in X$, then the limit

$$Q(x) := \lim_{n} 4^{sn} f(2^{-sn}x) \tag{3.6}$$

exists for all $x \in X$, and $Q : X \to Y$ is a unique quadratic function satisfying (1.6), *and*

$$\|f(x) - Q(x)\| \le \frac{1}{8}\sum_{i=(s+1)/2}^{\infty} 4^{si}\left(\phi\left(2^{-si}x, 2^{-si}x\right) + \frac{1}{2}\phi\left(0, 2^{-si}x\right)\right), \tag{3.7}$$

for all $x \in X$.

Proof. Let $s = 1$. Putting $x = 0$ in (3.3), we get

$$\|2[f(3y) - 3f(2y) + 3f(y)]\| \le \phi(0,y), \tag{3.8}$$

for all $y \in X$. On the other hand by replacing y by x in (3.3), it follows that

$$\|-f(3y) + 4f(2y) - 7f(y)\| \le \phi(y,y), \tag{3.9}$$

for all $y \in X$. Combining (3.8) and (3.9), we lead to

$$\|2f(2y) - 8f(y)\| \le 2\phi(y,y) + \phi(0,y), \tag{3.10}$$

for all $y \in X$. With the substitution $y := x/2$ in (3.10) and then dividing both sides of inequality by 2, we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \le \frac{1}{2}\left[2\phi\left(\frac{x}{2}, \frac{x}{2}\right) + \phi\left(0, \frac{x}{2}\right)\right]. \tag{3.11}$$

Now, using methods similar as in [8, 34, 35], we can easily show that the function $Q : X \to Y$ defined by $Q(x) = \lim_{n\to\infty} 4^n f(x/2^n)$ for all $x \in X$ is unique quadratic function satisfying (1.6) and (3.7). Let $s = -1$. Then by (3.10) we have

$$\left\| \frac{f(2x)}{4} - f(x) \right\| \le \frac{1}{8}(2\phi(x,x) + \phi(0,x)), \tag{3.12}$$

for all $x \in X$. And analogously, as in the case $s = 1$, we can show that the function $Q : X \to Y$ defined by $Q(x) := \lim_{n\to\infty} 4^{-n} f(2^n x)$ is unique quadratic function satisfying (1.6) and (3.7). □

Theorem 3.2. *Let $s \in \{1, -1\}$ be fixed. Let $\phi : X \times X \to [0, \infty)$ is a mapping such that*

$$\sum_{i=1}^{\infty} 2^{si} \left[\phi\left(\frac{x}{2^{si}}, \frac{x}{2^{si+1}}\right) + \phi\left(0, \frac{x}{2^{si+1}}\right)\right] < \infty \tag{3.13}$$

and that

$$\lim_{n\to\infty} 2^{sn} \phi\left(\frac{x}{2^{sn}}, \frac{y}{2^{sn}}\right) = 0, \tag{3.14}$$

for all $x, y \in X$.

Suppose that an odd mapping $f : X \to Y$ satisfies

$$\|D_f(x,y)\| \le \phi(x,y), \tag{3.15}$$

for all $x, y \in X$.

Then the limit

$$A(x) := \lim_{n\to\infty} 2^{sn} \left[f\left(\frac{x}{2^{sn-1}}\right) - 8f\left(\frac{x}{2^{sn}}\right)\right] \tag{3.16}$$

exists, for all $x \in X$, and $A : X \to Y$ is a unique additive function satisfying (1.6), *and*

$$\|f(2x) - 8f(x) - A(x)\| \le \sum_{i=|s-1|/2}^{\infty} 2^{si} \phi\left(\frac{x}{2^{si}}, \frac{x}{2^{si+1}}\right) + 2 \sum_{i=|s-1|/2}^{\infty} 2^{si} \phi\left(0, \frac{x}{2^{si+1}}\right), \tag{3.17}$$

for all $x \in X$.

Proof. Let $s = 1$. set $x = 0$ in (3.15). Then by oddness of f we have

$$\|2f(3y) - 8f(2y) + 16f(y)\| \leq \phi(0, y), \tag{3.18}$$

for all $y \in X$. Replacing x by $2y$ in (3.15) we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \leq \phi(2y, y). \tag{3.19}$$

Combining (3.18) and (3.19), we lead to

$$\|f(4y) - 10f(2y) + 16f(y)\| \leq \phi(2y, y) + 2\phi(0, y), \tag{3.20}$$

for all $y \in X$. Putting $y := x/2$ and $g(x) := f(2x) - 8f(x)$, for all $x \in X$. Then we get

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| \leq \phi\left(x, \frac{x}{2}\right) + 2\phi\left(0, \frac{x}{2}\right), \tag{3.21}$$

for all $x \in X$. Now, in a similar way as in [8, 34, 35], we can show that the limit $A(x) := \lim_{n\to\infty} 2^n g(x/2^n)$ exists, for all $x \in X$, and A is the unique function satisfying (1.6) and (3.17). If $s = -1$, then the proof is analogous. □

Theorem 3.3. *Let $s \in \{1, -1\}$ be fixed. Suppose that an odd mapping $f : X \to Y$ satisfies*

$$\|D_f(x, y)\| \leq \phi(x, y), \tag{3.22}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that

$$\sum_{i=1}^{\infty} 8^{si}\phi\left(\frac{x}{2^{si}}, \frac{x}{2^{si+1}}\right) + \sum_{i=1}^{\infty} 8^{si}\phi\left(0, \frac{x}{2^{si+1}}\right) < \infty \tag{3.23}$$

and that $\lim_{n\to\infty} 8^{sn}\phi(x/2^{sn}, y/2^{sn}) = 0$, for all $x, y \in X$, then the limit

$$C(x) := \lim_{n\to\infty} 8^{sn}\left[f\left(\frac{x}{2^{sn-1}}\right) - 2f\left(\frac{x}{2^{sn}}\right)\right] \tag{3.24}$$

exists, for all $x \in X$, and $C : X \to Y$ is a unique cubic function satisfying (1.6) *and*

$$\|f(2x) - 2f(x) - C(x)\| \leq \sum_{i=|s-1|/2}^{\infty} 8^{si}\phi\left(\frac{x}{2^{si}}, \frac{x}{2^{si+1}}\right) + 2\sum_{i=|s-1|/2}^{\infty} 8^{si}\phi\left(0, \frac{x}{2^{si+1}}\right), \tag{3.25}$$

for all $x \in X$.

Proof. We prove the theorem for $s = 1$. When $s = -1$ we have a similar proof. It is easy to see that f satisfies (3.20). Set $h(x) := f(2x) - 2f(x)$ then by putting $y := x/2$ in (3.20), it follows that

$$\left\| h(x) - 8h\left(\frac{x}{2}\right) \right\| \leq \phi\left(x, \frac{x}{2}\right) + 2\phi\left(0, \frac{x}{2}\right), \tag{3.26}$$

for all $x \in X$. By using (3.26), we may define a mapping $C : X \to Y$ as $C(x) := \lim_{n\to\infty} 8^n h(x/2^n)$, for all $x \in X$. Similar to Theorem 3.1, we can show that C is the unique cubic function satisfying (1.6) and (3.25). □

Theorem 3.4. *Suppose that an odd mapping $f : X \to Y$ satisfies*

$$\|D_f(x, y)\| \leq \phi(x, y), \tag{3.27}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that

$$\sum_{i=1}^{\infty} 8^i \phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \sum_{i=1}^{\infty} 8^i \phi\left(0, \frac{x}{2^{i+1}}\right) < \infty, \tag{3.28}$$

and that $\lim_{n\to\infty} 8^n \phi(x/2^n, y/2^n) = 0$, for all $x, y \in X$, then there exists a unique cubic function $C : X \to Y$ and a unique additive function $A : X \to Y$ such that

$$\|f(x) - C(x) - A(x)\| \leq \frac{1}{6}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{1}{3}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(0, \frac{x}{2^{i+1}}\right), \tag{3.29}$$

for all $x \in X$.

Proof. By Theorems 3.2 and 3.3, there exist an additive mapping $A_o : X \to Y$ and a cubic mapping $C_o : X \to Y$ such that

$$\begin{aligned} \|f(2x) - 8f(x) - A_o(x)\| &\leq \sum_{i=|s-1|/2}^{\infty} 2^{si}\phi\left(\frac{x}{2^{si}}, \frac{x}{2^{si+1}}\right) + 2\sum_{i=|s-1|/2}^{\infty} 2^{si}\phi\left(0, \frac{x}{2^{si+1}}\right), \\ \|f(2x) - 2f(x) - C_o(x)\| &\leq \sum_{i=|s-1|/2}^{\infty} 8^{si}\phi\left(\frac{x}{2^{si}}, \frac{x}{2^{si+1}}\right) + 2\sum_{i=|s-1|/2}^{\infty} 8^{si}\phi\left(0, \frac{x}{2^{si+1}}\right), \end{aligned} \tag{3.30}$$

for all $x \in X$. Combine the two equations of (3.30) to obtain

$$\left\| f(x) - \frac{1}{6}C_o(x) + \frac{1}{6}A_o(x) \right\| \leq \frac{1}{6}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{1}{3}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(0, \frac{x}{2^{i+1}}\right), \tag{3.31}$$

for all $x \in X$. So we get (3.29) by letting $A(x) = -(1/6)A_o(x)$, and $C(x) = (1/6)C_o(x)$, for all $x \in X$. To prove the uniqueness of A and C, let $A_1, C_1 : X \to Y$ be another additive and cubic maps satisfying (3.29). Let $A' = A - A_1$, and let $C' = C - C_1$. So

$$\begin{aligned}\|A'(x) - C'(x)\| &\le \|f(x) - A(x) - C(x)\| + \|f(x) - A_1(x) - C_1(x)\| \\ &\le 2\left[\frac{1}{30}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(\frac{x}{2^i}, \frac{x}{2^{i+1}}\right) + \frac{1}{15}\sum_{i=0}^{\infty}\left(2^i + 8^i\right)\phi\left(0, \frac{x}{2^{i+1}}\right)\right],\end{aligned} \tag{3.32}$$

for all $x \in X$. Since

$$\lim_{n\to\infty}\left\{\sum_{i=1}^{\infty}8^{i+n}\phi\left(\frac{x}{2^{i+n}}, \frac{x}{2^{i+n+1}}\right) + \sum_{i=1}^{\infty}8^{i+n}\phi\left(0, \frac{x}{2^{i+n+1}}\right)\right\} = 0, \tag{3.33}$$

then

$$\lim_{n\to\infty}\left\{\sum_{i=1}^{\infty}2^{i+n}\phi\left(\frac{x}{2^{i+n}}, \frac{x}{2^{i+n+1}}\right) + \sum_{i=1}^{\infty}2^{i+n}\phi\left(0, \frac{x}{2^{i+n+1}}\right)\right\} = 0, \tag{3.34}$$

for all $x \in X$. Hence (3.32) implies that

$$\lim_{n\to\infty} 8^n\left\|A'\left(\frac{x}{2^n}\right) - C'\left(\frac{x}{2^n}\right)\right\| = 0, \tag{3.35}$$

for all $x \in X$. On the other hand C and C_1 are cubic, then $C'(x/2^n) = (1/8^n)C'(x)$. Therefore by (3.35) we obtain that $A'(x) = 0$, for all $x \in X$. Again by (3.35) we have $C'(x) = 0$, for all $x \in X$. □

Theorem 3.5. *Suppose that an odd mapping $f : X \to Y$ satisfies*

$$\|D_f(x, y)\| \le \phi(x, y), \tag{3.36}$$

for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0, \infty)$ is a mapping such that

$$\sum_{i=1}^{\infty}\frac{1}{2^i}\phi\left(2^i x, 2^{i-1}x\right) + \sum_{i=1}^{\infty}2^i\phi\left(0, 2^{i-1}x\right) < \infty \tag{3.37}$$

and that $\lim_{n\to\infty}(1/2^n)\phi(2^n x, 2^n y) = 0$, for all $x, y \in X$, then there exist a unique cubic function $C : X \to Y$ and a unique additive function $A : X \to Y$ such that

$$\begin{aligned}&\|f(x) - C(x) - A(x)\| \\ &\quad\le \frac{1}{30}\sum_{i=1}^{\infty}\left(\frac{1}{2^i} + \frac{1}{8^i}\right)\left(\phi\left(2^i x, 2^{i-1}x\right)\right) + \frac{1}{15}\sum_{i=1}^{\infty}\left(\frac{1}{2^i} + \frac{1}{8^i}\right)\left(\phi\left(0, 2^{i-1}x\right)\right),\end{aligned} \tag{3.38}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.4. □

Now we establish the generalized Hyers-Ulam-Rassias stability of functional equation (1.6) as follows.

Theorem 3.6. *Suppose that a mapping $f : X \to Y$ satisfies $f(0) = 0$ and $\|D_f(x,y)\| \le \phi(x,y)$, for all $x, y \in X$. If the upper bound $\phi : X \times X \to [0,\infty)$ is a mapping such that*

$$\sum_{i=0}^{\infty}\left\{8^i\left[\phi\left(\frac{x}{2^i},\frac{x}{2^{i+1}}\right)+\phi\left(0,\frac{x}{2^{i+1}}\right)\right]+4^i\phi\left(\frac{x}{2^i},\frac{x}{2^i}\right)\right\}<\infty \tag{3.39}$$

and that $\lim_{n\to\infty}8^n\phi(x/2^n, y/2^n) = 0$, for all $x, y \in X$, then there exist a unique additive function $A : X \to Y$ a unique quadratic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\begin{aligned}&\|f(x)-A(x)-Q(x)-C(x)\|\\&\quad\le\frac{1}{6}\sum_{i=0}^{\infty}\left(2^i+8^i\right)\left[\phi\left(\frac{x}{2^i},\frac{x}{2^{i+1}}\right)+2\phi\left(0,\frac{x}{2^{i+1}}\right)\right]+\frac{1}{8}\sum_{i=1}^{\infty}4^i\left[\phi\left(\frac{x}{2^i},\frac{x}{2^i}\right)+\frac{1}{2}\phi\left(0,\frac{x}{2^i}\right)\right],\end{aligned} \tag{3.40}$$

for all $x \in X$.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$, for all $x \in X$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$, and $\|D_{f_e}(x,y)\| \le (1/2)[\phi(x,y) + \phi(-x,-y)]$, for all $x, y \in X$. Hence in view of Theorem 3.1 there exists a unique quadratic function $Q : X \to Y$ satisfying (3.7). Let $f_o(x) = (1/2)(f(x) - f(-x))$, for all $x \in X$. Then $f_o(0) = 0, f_o(-x) = -f_o(x)$, and $\|D_{f_o}(x,y)\| \le (1/2)[\phi(x,y) + \phi(-x,-y)]$, for all $x, y \in X$. From Theorem 3.4, it follows that there exist a unique cubic function $C : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (3.29). Now it is obvious that (3.40) holds true for all $x \in X$, and the proof of theorem is complete. □

Corollary 3.7. *Let $p + q > 3$, $\theta \ge 0$. Suppose that a mapping $f : X \to Y$ satisfies $f(0) = 0$, and*

$$\|D_f(x,y)\| \le \theta\left(\|x\|^p\|y\|^q\right), \tag{3.41}$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$, a unique quadratic function $Q : X \to Y$, and a unique cubic function $C : X \to Y$ satisfying

$$\|f(x)-A(x)-Q(x)-C(x)\|\le\theta\|x\|^{p+q}\left[\left(\frac{1}{6\times2^q}\right)\left(\frac{2}{2-2^{p+q}}+\frac{8}{8-2^{p+q}}\right)+\frac{1}{8}\left(\frac{2^{p+q}}{4-2^{p+q}}\right)\right], \tag{3.42}$$

for all $x \in X$.

Proof. It follows from Theorem 3.6 by taking $\phi(x,y) = \theta(\|x\|^p\|y\|^q)$, for all $x, y \in X$. □

Theorem 3.8. *Suppose that* $f : X \to Y$ *satisfies* $f(0) = 0$, *and* $\|D_f(x,y)\| \le \phi(x,y)$, *for all* $x, y \in X$. *If the upper bound* $\phi : X \times X \to [0,\infty)$ *is a mapping such that*

$$\sum_{i=1}^{\infty}\left\{\frac{1}{2^i}\left[\phi\left(2^i x, 2^{i-1}x\right)+\phi\left(0,2^{i-1}x\right)\right]+\frac{1}{4^i}\phi\left(2^i x,2^i x\right)\right\}<\infty \tag{3.43}$$

and that $\lim_{n\to\infty}(1/2^n)\phi(2^n x, 2^n y) = 0$, *for all* $x, y \in X$, *then there exists a unique additive function* $A : X \to Y$, *a unique quadratic function* $Q : X \to Y$, *and a unique cubic function* $C : X \to Y$ *such that*

$$\begin{aligned}&\|f(x)-A(x)-Q(x)-C(x)\|\\&\quad\le\frac{1}{6}\left[\sum_{i=1}^{\infty}\left(\frac{1}{2^i}+\frac{1}{8^i}\right)\left(\phi\left(2^i x,2^{i-1}x\right)+2\phi\left(0,2^{i-1}x\right)\right)\right]+\frac{1}{8}\sum_{i=0}^{\infty}\frac{1}{4^i}\left[\phi\left(2^i x,2^i x\right)+\frac{1}{2}\phi\left(0,2^i x\right)\right],\end{aligned} \tag{3.44}$$

for all $x \in X$.

By Theorem 3.8, we are going to investigate the following stability problem for functional equation (1.6).

Corollary 3.9. *Let* $p + q < 1$, $\theta > 0$. *Suppose that* $f : X \to Y$ *satisfies* $f(0) = 0$, *and*

$$\|D_f(x,y)\| \le \theta(\|x\|^p\|y\|^q), \tag{3.45}$$

for all $x, y \in X$, *then there exist a unique additive function* $A : X \to Y$, *a unique quadratic function* $Q : X \to Y$, *and a unique cubic function* $C : X \to Y$ *satisfying*

$$\begin{aligned}&\|f(x)-A(x)-Q(x)-C(x)\|\\&\quad\le\theta\|x\|^{p+q}\left\{\left(\frac{1}{6\times 2^q}\right)\left(\frac{2^{p+q}}{2-2^{p+q}}+\frac{2^{p+q}}{8-2^{p+q}}\right)+\frac{1}{8-2^{p+q+3}}\right\},\end{aligned} \tag{3.46}$$

for all $x \in X$.

By Corollary 3.9, we solve the following Hyers-Ulam stability problem for functional equation (1.6).

Corollary 3.10. *Let* ϵ *be a positive real number. Suppose that a mapping* $f : X \to Y$ *satisfies* $f(0) = 0$, *and* $\|D_f(x,y)\| \le \epsilon$, *for all* $x, y \in X$, *then there exist a unique additive function* $A : X \to Y$, *a unique quadratic function* $Q : X \to Y$, *and a unique cubic function* $C : X \to Y$ *such that*

$$\|f(x)-A(x)-Q(x)-C(x)\| \le \frac{5}{14}\epsilon, \tag{3.47}$$

for all $x \in X$.

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