RATE OF CONVERGENCE OF SOLUTIONS OF RATIONAL DIFFERENCE EQUATION OF SECOND ORDER

S. KALABUŠIĆ AND M. R. S. KULENOVIĆ

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We investigate the rate of convergence of solutions of some special cases of the equation $x_{n+1} = (\alpha+\beta x_n+\gamma x_{n-1})/(A+Bx_n+Cx_{n-1})$, $n = 0,1,\ldots,$ with positive parameters and nonnegative initial conditions. We give precise results about the rate of convergence of the solutions that converge to the equilibrium or period-two solution by using Poincaré's theorem and an improvement of Perron's theorem.

1. Introduction and preliminaries

We investigate the rate of convergence of solutions of some special types of the second-order rational difference equation

$$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}, \quad n = 0,1,\ldots, \tag{1.1}$$

where the parameters α, β, γ, A, B, and C are positive real numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative real numbers.

Related nonlinear second-order rational difference equations were investigated in [2, 5, 6, 7, 8, 9, 10]. The study of these equations is quite challenging and is in rapid development.

In this paper, we will demonstrate the use of Poincaré's theorem and an improvement of Perron's theorem to determine the precise asymptotics of solutions that converge to the equilibrium.

We will concentrate on three special cases of (1.1), namely, for $n = 0,1,\ldots,$

$$x_{n+1} = \frac{B}{x_n} + \frac{C}{x_{n-1}}, \tag{1.2}$$

$$x_{n+1} = \frac{px_n+x_{n-1}}{qx_n+x_{n-1}}, \tag{1.3}$$

$$x_{n+1} = \frac{px_n+x_{n-1}}{q+x_{n-1}}, \tag{1.4}$$

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(e) *A necessary and sufficient condition for one root of (1.8) to have an absolute value greater than one and for the other to have an absolute value less than one is*

$$s^2 + 4t > 0, \quad |s| > |1 - t|. \tag{1.11}$$

In this case, the unstable equilibrium $\bar{x}$ is called a saddle point.

The set of points whose orbits converge to an attracting equilibrium point or, periodic point is called the "basin of attraction," see [1, page 128].

Definition 1.2. Let $\mathbf{T}$ be a map on $\mathbb{R}^2$ and let $\mathbf{p}$ be an equilibrium point or a periodic point for $\mathbf{T}$. The *basin of attraction* of $\mathbf{p}$, denoted by $\mathcal{B}_{\mathbf{p}}$, is the set of points $\mathbf{x} \in \mathbb{R}^2$ such that $|\mathbf{T}^k(\mathbf{x}) - \mathbf{T}^k(\mathbf{p})| \to 0$, as $k \to \infty$, that is,

$$\mathcal{B}_{\mathbf{p}} = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{T}^k(\mathbf{x}) - \mathbf{T}^k(\mathbf{p})| \longrightarrow 0, \text{ as } k \longrightarrow \infty\}, \tag{1.12}$$

where $|\cdot|$ denotes any norm in $\mathbb{R}^2$.

We now give the definitions of positive and negative semicycles of a solution of (1.5) relative to an equilibrium point $\bar{x}$.

A *positive semicycle* of a solution $\{x_n\}$ of (1.5) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that either $l = -1$ or $l > -1$, $x_{l-1} < \overline{x}$, and either $m = \infty$ or $m < \infty$, $x_{m+1} < \overline{x}$. A *negative semicycle* of a solution $\{x_n\}$ of (1.5) consists of a string of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than the equilibrium $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that either $l = -1$ or $l > -1, x_{l-1} \geq \overline{x}$, and either $m = \infty$ or $m < \infty, x_{m+1} \geq \overline{x}$.

The next theorem is a slight modification of the result obtained in [7, 9].

Theorem 1.3. *Assume that*

$$f : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \tag{1.13}$$

is a continuous function satisfying the following properties:

(a) *there exist L and U, $0 < L < U$, such that*

$$f(L,L) \geq L, \qquad f(U,U) \leq U, \tag{1.14}$$

and $f(x,y)$ is nondecreasing in x and y in $[L,U]$;

(b) *the equation*

$$f(x,x) = x \tag{1.15}$$

has a unique positive solution in $[L,U]$.

Then (1.5) has a unique equilibrium $\overline{x} \in [L,U]$ and every solution of (1.5) with initial values $x_{-1}, x_0 \in [L,U]$ converges to $\overline{x}$.