Hölder Continuity for the Displacements in Isotropic and Kinematic Hardening with von Mises Yield Criterion

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Abstract

We consider the regularity of weak solutions to evolution variational inequalities arising from the flow theory of plasticity with isotropic and kinematic hardening. The (linear) elasticity tensor is allowed to have discontinuities.

We derive a Morrey condition for the stress velocities and the strains (not the strain velocity!) up to the boundary. In the case of two space dimensions we conclude the Hölder continuity of the displacements.

Keywords: plasticity with hardening, isotropic & kinematic hardening, Hölder continuity, von Mises yield criterion

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1 Introduction and formulation of the problem

In this paper we deal with solutions of the flowrule of plasticity with hardening. This means we look for solutions $((\sigma,\xi),v) \in (\mathcal{M}\cap\mathcal{K}) \times L^1(0,T;BD(\Omega))$ and that for all $(\tau,\eta) \in \mathcal{M}\cap\mathcal{K}$ holds

$$\big(A\dot{\sigma},\tau-\sigma\big) + \big(\dot{\xi},\eta-\xi\big) + \langle v, \operatorname{div}(\tau-\sigma)\rangle \le 0 \tag{1.1a}$$

$$\big(\sigma,\nabla w\big) = \big(f,w\big) + \int_{\Gamma_N} pw do \quad \forall\, w \in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n) \text{ a.e with respect to } t \tag{1.1b}$$

$$\begin{aligned}(\sigma,\xi)(0) = 0 \text{ in } \overline{\Omega}\times\mathbb{R}^m\times\{t=0\}\\ v = 0 \text{ on } \Gamma_D\times[0,T]\end{aligned} \tag{1.2}$$

Remark A choice of $\sigma(0)=\sigma_0$ sufficiently smooth is also possible.

Here

$$\mathcal{K} = \big\{(\tau,\eta)\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))\times L^2(0,T;L^2(\Omega,\mathbb{R}))\,|\,F(\tau,\eta)\le 0 \text{ a.e. in } \Omega\times[0,T]\big\}$$

$$\mathcal{M} = \big\{(\tau,\eta)\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))\times L^2(0,T;L^2(\Omega,\mathbb{R}))\,|\,\operatorname{div}\tau\in L^\infty(0,T;L^n(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))\big\}$$

$F : \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^m \to \mathbb{R}$ is a given convex continuous function, called yield function. For the definition of the usual function spaces $L^a(0,T;L^b)$ cf. [DL76] or the space $BD(\Omega)$ of functions with bounded deformation see [Tem85].

We suppose the usual **safe load condition** (cf. Johnson [Joh78]). There exists an element $(\sigma^0,\xi^0) \in W^{1,\infty}\big(0,T;L^\infty(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}) \times L^\infty(\Omega,\mathbb{R}^n)\big)$ such that

$$\begin{aligned} F(\sigma^0,\xi^0) &\leq -\delta_0 < 0 \\ -\operatorname{div}\sigma^0 &= f \text{ in } \Omega\times[0,T] \\ \sigma^0\cdot\vec{n} &= p \text{ on } \Gamma_N\times[0,T] \\ (\sigma^0,\xi^0)(0) &= 0 \text{ in } \Omega\times\{t=0\} \end{aligned} \tag{1.3}$$

Let $\Omega\subset\mathbb{R}^n$, $n\geq 2$ be an open bounded and connected subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega = \Gamma_D\dot{\cup}\Gamma_N$.
Let

$$f,\dot{f} \in L^\infty(0,T;L^n(\Omega,\mathbb{R}^n)) \tag{1.4a}$$

$$p,\dot{p} \in L^\infty(0,T;L^\infty(\partial\Omega,\mathbb{R}^n)) \tag{1.4b}$$

and set $\varepsilon(u) = \frac{1}{2}(Du + Du^\mathsf{T})$. For the sake of simplicity we do not require the optimal function spaces for proving our regularity results. Let $BD(\Omega)$ denote the space of functions with bounded deformation, that is

$$BD(\Omega) = \big\{u\in L^1(\Omega,\mathbb{R}^n)\,|\,\varepsilon(u)\in\big(C_0(\Omega,\mathbb{R}^n)\big)^*\big\}.$$

This means the strain tensor $\varepsilon(u)$ is only a bounded measure and there is a continuous embedding $BD(\Omega)\hookrightarrow L^{\frac{n}{n-1}}$.

Let $A\in L^\infty(\Omega;\hom(\mathbb{R}^{n\times n}_{\text{sym}},\mathbb{R}^{n\times n}_{\text{sym}}))$ be a uniform elliptic, symmetric fourth order tensor with ellipticity constant $\alpha_A > 0$. The tensor A describes the elastic material properties (for example the inverse Lamé-Navier-Operator).
The parameter t is a loading parameter and $v = \dot{u}$ is interpreted as the displacement velocity, u as the displacement field of the body Ω. The quantity σ is the stress tensor and the function $\xi : \Omega\to\mathbb{R}^m$ is the so called hardening parameter. (here $m = 1$ or $m = n\times n$)

In this paper we consider the case of the von Mises yield criterion for isotropic and kinematic hardening.

$$F(\sigma,\xi) = |\sigma_D| - (\xi+\kappa) \qquad \text{isotropic}$$

$$F(\sigma,\xi) = |\sigma_D - \xi_D| - \kappa \qquad \text{kinematic}$$

with $\kappa > 0$ and $\sigma_D = \sigma - \frac{1}{n}\,\mathrm{tr}(\sigma) Id$ the deviator of σ.

Under the assumption of the safe load condition (1.3) and (1.4), it is known that the above problem has a solution $((\sigma,\xi),v)$ [Joh78] and in addition it is known that $v \in L^2(0,T;H^1(\Omega,\mathbb{R}^n))$ [Joh78] and $(\sigma,\xi) \in L^\infty(0,T;H^1_{loc}(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}})) \times L^\infty(0,T;H^1_{loc}(\Omega))$ [Ser94][Löb07]. (For the differentiability of (σ,ξ), one has to assume $A \in W^{1,\infty}$ rather than L^∞.)

We empasize, that the coefficients of A are *only* in L^∞. If A is smooth, say Lipschitz, Seregin [Ser94] proved in the case of kinematic hardening, that $u \in L^\infty(0,T;H^2(\Omega,\mathbb{R}^n))$, this implies that $u \in C^{0,1/2}$ ($C^{0,1-\delta}$ for $n=2$) in space direction for three dimensions. In the case of isotropic hardening Seregin obtains only $\nabla u \in BD(\Omega)$, which does not imply Hölder continuity.

Let us remark that in case of Lipschitz coefficients of A, we have a proof that $u \in C^{0,1/2}$ $(n=3)$ (or $u \in C^{0,1-\delta}$ $(n=2)$) also in the case of isotropic hardening, but this is not the subject of the present paper.

The purpose of this paper is to prove a *Morrey condition* of the type

$$\int_0^T \int_{B_R} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \mathrm{d}x\,\mathrm{d}t \le 2^\alpha \frac{R^\alpha}{R_0^\alpha} \int_0^T \int_{B_{R_0}} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \mathrm{d}x\,\mathrm{d}t$$

up to the boundary of the basic domain Ω. In the case of 2 space dimensions one can conclude the everywhere continuity of the displacements $u(x,t) = \int_0^t v(x,s)\,\mathrm{d}s$.

2 Approximation

We now approximate the plasticity problem (1.1) with a viscoplastic type model.

Let $F(\sigma,\xi)$ denote the yield function. We define for $\mu > 0$ a viscoplastic type potential $G_\mu(\sigma,\xi)$.

$$G_\mu(\sigma,\xi) := \frac{1}{2\mu}\big(F(\sigma,\xi)\big)_+^2$$

where $(a)_+ = \begin{cases} a & \text{if } a \ge 0 \\ 0 & \text{if } a < 0 \end{cases}$

The term $G_\mu(\sigma,\xi)$ is almost everywhere differentiable and convex, thus it's derivative $G'_\mu(\sigma,\xi)$ is a monotone operator. For the von Mises yield criterion in the case of isotropic hardening we have

$$G_\mu(\sigma,\xi) = \frac{1}{2\mu}(|\sigma_D| - (\kappa+\xi))_+^2$$
$$G'_\mu(\sigma,\xi) = \frac{1}{\mu}(|\sigma_D| - (\kappa+\xi))_+ \cdot \begin{pmatrix} \frac{\sigma_D}{|\sigma_D|} \\ -1 \end{pmatrix}.$$

In the case of kinematic hardening

$$G_\mu(\sigma,\xi) = \frac{1}{2\mu}(|\sigma_D - \xi_D| - \kappa)_+^2$$

$$G'_\mu(\sigma,\xi) = \frac{1}{\mu}(|\sigma_D - \xi_D| - \kappa)_+ \frac{\sigma_D - \xi_D}{|\sigma_D - \xi_D|} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

The penalized hardening problem now reads:

Find $((\sigma_\mu, \xi_\mu), v_\mu) \in \mathcal{M} \times L^1(0,T; BD(\Omega))$ such that for all $(\tau,\eta) \in \mathcal{M},\ \tau \cdot \vec{n}$ on Γ_N holds:

$$\big(A\dot{\sigma}_\mu,\tau\big) + \big(\xi_\mu,\eta\big) + \big(G'_\mu((\sigma_\mu,\xi_\mu)),(\tau,\eta)\big) + \langle v_\mu, \operatorname{div}\tau\rangle = 0 \tag{2.1}$$

$$\begin{aligned} \big(\sigma_\mu,\nabla w\big) &= \big(f,w\big) + \int_{\Gamma_N} pwdo \;\forall\, w \in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n) \text{ a.e with respect to } t \\ (\sigma_\mu,\xi_\mu)(0) &= 0 \\ v_\mu &= 0 \text{ on } \Gamma_D \times t \end{aligned} \tag{2.2}$$

The existence of solutions and the convergence as $\mu \to 0$, to the solution of the initial hardening problem (1.1) can be found in [Löb07].

Moreover, for the approximations, we have the following estimates independent of the viscosity coefficient μ.

$$\|\sigma_\mu\|_{L^\infty(L^2)},\ \|\xi_\mu\|_{L^\infty(L^2)} \le Const$$

$$\|\dot{\sigma}_\mu\|_{L^2(L^2)},\ \|\dot{\xi}_\mu\|_{L^2(L^2)} \le Const$$

For fixed viscosity coefficient μ, we have $\varepsilon(v_\mu) \in L^2(0,T; L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ and the pointwise penalized hardening problem (both isotropic and kinematic)

$$\begin{pmatrix} A\dot{\sigma}_\mu \\ \dot{\xi}_\mu \end{pmatrix} + \begin{pmatrix} G'_\mu(\sigma_\mu,\xi_\mu)_1 \\ G'_\mu(\sigma_\mu,\xi_\mu)_2 \end{pmatrix} = \begin{pmatrix} \varepsilon(v_\mu) \\ 0 \end{pmatrix} \tag{2.3}$$

holds almost everywhere in $\Omega \times [0,T]$.

3 The tube-filling condition (interior case)

For the sake of clarity we omit the subscript μ for the viscosity coefficient.

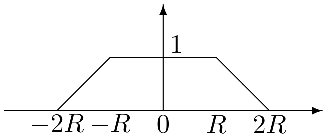
Theorem 1 *Let $((\sigma,\xi),v)$ be a solution of the penalized problem (2.1). Then for any $B_R = B_R(x_0) \subset B_{2R} = B_{2R}(x_o) \subset \Omega$ and $T > 0$ there holds the tube-filling condition*

$$\int_0^T \int_{B_{2R}} \big[|\nabla v|^2 + |\dot{\sigma}|^2 + |\dot{\xi}|^2\big]\,\mathrm{d}x\,\mathrm{d}t \le C \int_0^T \int_{B_{2R}\setminus B_R} \big[|\nabla v|^2 + |\dot{\sigma}|^2 + |\dot{\xi}|^2\big]\,\mathrm{d}x\,\mathrm{d}t + KR \tag{3.1}$$

where C, K are constants which are independent of the penalty parameter as $\mu \to 0$.

Proof Let ζ denote the usual localization function defined by $\zeta(x) = \zeta_0(|x_0 - x|)$ where

$$\zeta_0(s) = \begin{cases} 1 & s \in [-R, R] \\ \frac{2R-|s|}{R} & s \in [-2R, 2R] \setminus [-R, R] \\ 0 & s \notin [-2R, 2R]\,. \end{cases}$$



In equation (2.3), we choose the test function $\zeta^2(\dot\sigma, \dot\xi)^\intercal$ and obtain

$$\int_0^T \int_{B_{2R}} \varepsilon(v) : \zeta^2 \dot\sigma \,\mathrm{d}x\,\mathrm{dt} = \int_0^T \int_{B_{2R}} A\dot\sigma : \zeta^2 \dot\sigma \,\mathrm{d}x\,\mathrm{dt} + \int_0^T \int_{B_{2R}} \zeta^2 \dot\xi^2 \,\mathrm{d}x\,\mathrm{dt} + \int_{B_{2R}} \zeta^2 G_\mu(\sigma, \xi)(T)\,\mathrm{d}x\,. \tag{3.2}$$

The term arising by the penalty term is non-negative, thus using also the definiteness of the elastic material tensor A we obtain

$$c_0 \int_0^T \int_{B_{2R}} |\dot\sigma|^2 \zeta^2 \,\mathrm{d}x\,\mathrm{dt} + \int_0^T \int_{B_{2R}} |\dot\xi|^2 \zeta^2 \,\mathrm{d}x\,\mathrm{dt} \le \int_0^T \int_{B_{2R}} \varepsilon(v_\mu) : \dot\sigma \zeta^2 \,\mathrm{d}x\,\mathrm{dt} \tag{3.3}$$

We differentiate (1.1b) with respect to t and use this to rewrite the righthand side of (3.3)

$$\begin{aligned} \int_0^T \int_{B_{2R}} \varepsilon(v) : \zeta^2 \dot\sigma \,\mathrm{d}x\,\mathrm{dt} &= -\int_0^T \int_{B_{2R}} (v - \overline{v}_R) \dot f \zeta^2 \,\mathrm{d}x\,\mathrm{dt} - \int_0^T \int_{B_{2R}\setminus B_R} (v - \overline{v}_R) 2\zeta \dot\sigma \nabla\zeta \,\mathrm{d}x\,\mathrm{dt} \\ &:= E_1 + E_2 \end{aligned} \tag{3.4}$$

where $\overline{v}_R = \fint_{B_{2R}} v\,\mathrm{d}x\,.$

The boundary integral with p does not occur here. The Term E_1 can be estimated

$$\begin{aligned} |E_1| &\le \int_0^T K\|v\| \cdot \|\dot f \zeta^2\|\,\mathrm{dt} \\ &\le KR \int_0^T \|v\|\,\mathrm{dt} \le K(T)R\,. \end{aligned} \tag{3.5}$$

The term E_2 is split into

$$\begin{aligned} |E_2| &\le \frac{R^{-2}}{4a} \int_0^T \|(v - \overline{v}_R)\chi_{B_{2R}\setminus B_R}\|^2\,\mathrm{dt} + 2a \int_0^T \|\zeta\dot\sigma\|^2\,\mathrm{dt} \\ &\le \frac{K}{4a} \int_0^T \|\nabla v \chi_{B_{2R}\setminus B_R}\|^2\,\mathrm{dt} + 2a \int_0^T \|\zeta\dot\sigma\|^2\,\mathrm{dt} \end{aligned} \tag{3.6}$$

due to Poincarés inequality. We choose a small enough such that the terms with $\dot\sigma$ at the righthand side of (3.6) can be absorbed by the corresponding one in (3.3). We fix:

$$\int_0^T \int_{B_{2R}} \zeta^2 (|\dot\sigma|^2 + |\dot\xi|^2)\,\mathrm{d}x\,\mathrm{dt} \le K(T)R + K \int_0^T \|\nabla v \chi_{B_{2R}\setminus B_R}\|\,\mathrm{dt} \tag{3.7}$$

Now we arrange an estimate for $\iint |\varepsilon(v)|^2\zeta^2 \,\mathrm{d}x\,\mathrm{dt}$:

In the case of *isotropic hardening* we test the penalized equation (2.3) with $\psi := \zeta^2\big(\varepsilon(v), |\varepsilon(v)_D|\big)^{\mathsf{T}}$, and see that the penalty term is pointwise non positive

$$\mu G'_\mu(\sigma,\xi)\psi = (|\sigma_D| - (\kappa-\xi))_+\zeta^2\left(\frac{\sigma_D}{|\sigma_D|}\varepsilon(v) - |\varepsilon(v)_D|\right) \le 0$$

This yields

$$\int_0^T\int_{B_{2R}} |\zeta\varepsilon(v)|^2\,\mathrm{d}x\,\mathrm{dt} \le \int_0^T\int_{B_{2R}} \zeta^2(A\dot\sigma):\varepsilon(v)\,\mathrm{d}x\,\mathrm{dt} + \int_0^T\int_{B_{2R}} \zeta^2|\varepsilon(v)|\dot\xi\,\mathrm{d}x\,\mathrm{dt}\,.$$

In the case of *kinematic hardening* we choose the testfunction $\tilde\psi := \zeta^2\big(\varepsilon(v),\varepsilon(v)\big)^{\mathsf{T}}$ and obtain

$$\mu G'_\mu(\sigma,\xi)\tilde\psi = 0\,.$$

This yields

$$\int_0^T\int_{B_{2R}} |\zeta\varepsilon(v)|^2\,\mathrm{d}x\,\mathrm{dt} \le \int_0^T\int_{B_{2R}} \zeta^2(A\dot\sigma):\varepsilon(v)\,\mathrm{d}x\,\mathrm{dt} + \int_0^T\int_{B_{2R}} \zeta^2\varepsilon(v):\dot\xi\,\mathrm{d}x\,\mathrm{dt}\,.$$

Thus in both cases we obtain via Young's inequality:

$$\int_0^T\int_{B_{2R}} |\zeta\varepsilon(v)|^2\,\mathrm{d}x\,\mathrm{dt} \le C\int_0^T\int_{B_{2R}} |\zeta\dot\sigma|^2\,\mathrm{d}x\,\mathrm{dt} + C\int_0^T\int_{B_{2R}} |\zeta\dot\xi|^2\,\mathrm{d}x\,\mathrm{dt} \tag{3.8}$$

We rewrite this and estimate using Korn's and Young's inequality

$$\begin{aligned}\int_0^T\int_{B_{2R}} |\zeta\varepsilon(v)|^2\,\mathrm{d}x\,\mathrm{dt} \ge& \int_0^T\int_{B_{2R}} |\varepsilon(\zeta(v-c))|^2\,\mathrm{d}x\,\mathrm{dt} - K\int_0^T\int_{B_R} \zeta|\nabla\zeta||\nabla v||v-c|\,\mathrm{d}x\,\mathrm{dt}\\ &- K\int_0^T\int_{B_R} |\nabla\zeta|^2|v-c|^2\,\mathrm{d}x\,\mathrm{dt}\\ \ge& c_1\int_0^T\int_{B_R} |\nabla(\zeta(v-c))|^2\,\mathrm{d}x\,\mathrm{dt} - \frac{c_1}{2}\int_0^T\int_{B_{2R}} \zeta^2|\nabla v|^2\,\mathrm{d}x\,\mathrm{dt}\\ &- K\int_0^T\int_{B_R} |\nabla\zeta|^2|v-c|^2\,\mathrm{d}x\,\mathrm{dt}\,.\end{aligned} \tag{3.9}$$

We proceed by using Korn's and Poincaré's inequality again and obtain ($c = \frac{c_1}{2}$)

$$c\int_0^T\int_{B_{2R}} \zeta^2|\nabla v|^2\,\mathrm{d}x\,\mathrm{dt} - C\int_0^T\int_{B_{2R}\setminus B_R} |\nabla v|^2\,\mathrm{d}x\,\mathrm{dt} \le \int_0^T\int_{B_{2R}} |\zeta\varepsilon(v)|^2\,\mathrm{d}x\,\mathrm{dt}\,. \tag{3.10}$$

If we collect these estimates we obtain

$$\int_0^T\int_{B_R} \big[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\big]\,\mathrm{d}x\,\mathrm{dt} \le C\int_0^T\int_{B_{2R}\setminus B_R} \big[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\big]\,\mathrm{d}x\,\mathrm{dt} + KR \tag{3.11}$$

which is the statement of the theorem.

Note: That the terms $|\dot\sigma|^2 + |\dot\xi|^2$ on the right hand side of (3.11) are redundant, however if our proof is refined in order to improve the constants we would use these terms.

4 The tube-filling step and the Morrey condition

Theorem 2 *If the regularity assumptions* (1.4) *hold true and the elastic material tensor A is uniformly elliptic, there exists $\alpha \in (0,1)$ such that*

$$\int_0^T \int_{B_R} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \mathrm{d}x\,\mathrm{d}t \le 2^\alpha \frac{R^\alpha}{R_0^\alpha} \left[\int_0^T \int_{B_{R_0}} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \mathrm{d}x\,\mathrm{d}t + 2KR_0\right] \quad (4.1)$$

for all $B_R(x_0) \subset B_{\frac{R_0}{2}}(x_0) \subset B_{R_0}(x_0) \subset \Omega$. The estimate is independent of μ as $\mu \to 0$.

Remark: Due to the convergence $\sigma_\mu \to \sigma$, $\dot\sigma_\mu \rightharpoonup \dot\sigma$, $\nabla v_\mu \rightharpoonup \nabla v$, inequality (1.1a) gives in the limit $\mu \to 0$

$$\int_0^T \int_{B_R} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \mathrm{d}x\,\mathrm{d}t \le KR^\alpha$$

Proof We add in inequality (3.11) the quantity $\int_0^T \int_{B_R} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \mathrm{d}x\,\mathrm{d}t$, i.e. we fill the tube $[0;T] \times \left(B_{2R} \setminus B_R\right)$ and obtain

$$\int_0^T \int_{B_R} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \le \frac{c}{1+c} \int_0^T \int_{B_{2R}} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] + KR \quad (4.2)$$

By iteration we conclude with $R_j = R_0 2^{-j}$

$$\begin{aligned}\int_0^T \int_{B_{R_N}} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \le \left(\frac{c}{1+c}\right)^N &\left[\int_0^T \int_{B_{R_0}} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] + 2KR_0\right] \\ &+ K \sum_{\nu=1}^N R_\nu \left(\frac{c}{1+c}\right)^{N-\nu}\end{aligned} \quad (4.3)$$

choose α such that

$$\frac{c}{1+c} = 2^{-\alpha}\,. \quad (4.4)$$

Then (4.3) implies

$$\int_0^T \int_{B_{R_N}} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] \le \left(2^{-N}\right)^\alpha \left[\int_0^T \int_{B_{R_0}} \left[|\nabla v|^2 + |\dot\sigma|^2 + |\dot\xi|^2\right] + 2KR_0\right] \quad (4.5)$$

which is the statement for $R_0 = R_0 2^{-j}$. The case of general $R_{j+1} < R < R_j$ follows by estimating $R^{-\alpha} = (2R)^{-\alpha} 2^\alpha \le 2^\alpha R_j^{-\alpha}$.

5 Hölder continuity of the displacements in two dimensions

We are not able to prove the Hölder continuity of the displacement *velocities* v, however, we succeed this, for the displacements

$$u(x,t) = \int_0^t v(x,s)\,\mathrm{d}s + u_0(x) \quad (5.1)$$

Theorem 3 *Let $n = 2$, $u_0 \in C^{0,\alpha}$,and let the assumptions of theorem 2 hold true. Then the displacements u defined by* (5.1) *are Hölder continuous on interior domains. The Hölder exponent is given by $\alpha(2-\alpha)^{-1}$, where α comes from* (4.4)*. The estimates are independent of the penalty parameter μ.*

Remark: Again, due to the convergence we have the Hölder continuity of u in the limit $\mu \to 0$.

Proof The functions u are obviously Hölder continuous in spatial direction, i.e.

$$u \in L^\infty(0,T; C^{0,\alpha}(\Omega_0, \mathbb{R}^n))$$

for all $\Omega_0 \subset\subset \Omega$, since one easily estimates that $\nabla u \in L^\infty(0,T; L^2_{2-2\alpha}(\Omega, \mathbb{R}^{n\times n}))$.
$L^2_{2-2\alpha}$ denotes the usual Morrey space of functions satisfying $\int_{B_R} w^2 \,\mathrm{d}x \leq KR^{2\alpha}$, $(n=2)$.
We further have

$$\int_0^T \int_{B_R} \dot{u}^2 \,\mathrm{d}x \,\mathrm{dt} = \int_0^T \int_{B_R} v^2 \,\mathrm{d}x \,\mathrm{dt} \leq KR^{2\alpha}$$

and hence via Hölder's inequality,

$$\begin{aligned} \int_{B_R} |u(x,t_2) - u(x,t_1)|^2 \,\mathrm{d}x &\leq \int_{B_R} |t_2 - t_1| \int_{t_1}^{t_2} \dot{u}^2 \,\mathrm{dt} \,\mathrm{d}x \\ &\leq K|t_2 - t_1| R^{2\alpha} \,. \end{aligned} \tag{5.2}$$

From (5.2) we conclude

$$\left| \int_{B_R} u(x,t_2) \,\mathrm{d}x - \int_{B_R} u(x,t_1) \,\mathrm{d}x \right|^2 \leq |B_R| \int_{B_R} |u(x,t_2) - u(x,t_1)|^2 \,\mathrm{d}x$$

and

$$\begin{aligned} \left| ⨍_{B_R} u(x,t_2) \,\mathrm{d}x - ⨍_{B_R} u(x,t_1) \,\mathrm{d}x \right|^2 &\leq |B_R|^{-1} \int_{B_R} |u(x,t_2) - u(x,t_1)|^2 \,\mathrm{d}x \\ &\leq K|t_2 - t_1| R^{2\alpha - 2} \end{aligned} \tag{5.3}$$

Since $u \in L^\infty(0,T; C^{0,\alpha}(\Omega_0, \mathbb{R}^n))$ we know

$$\left| ⨍_{B_R} u(x,t_i) \,\mathrm{d}x - u(t_i, x_0) \right| \leq KR^\alpha$$

for $t_i = t_1, t_2$ almost everywhere with respect to t.
Then (5.3) implies

$$|u(x_0,t_2) - u(x_0,t_1)| \leq KR^\alpha + K|t_2 - t_1| R^{2\alpha-2}$$

Choosing $|t_2 - t_1| = R^{2-\alpha}$ we obtain

$$|u(x_0,t_2) - u(t_1,x_0)| \leq KR^\alpha = K|t_2 - t_1|^{\frac{\alpha}{2-\alpha}}$$

and we obtain a Hölder exponent $\beta = \frac{\alpha}{2-\alpha}$ with respect to the time. Since $|u(x_1,t) - u(x_2,t)| \leq K|x_1 - x_2|^{2\alpha}$ is already known, the theorem is proved.

6 The Morrey condition up to the boundary

For the boundary estimate we require a condition for the boundaries Γ_D and Γ_N .

Neumann boundary:

For $x_0 \in U(\partial\Omega \setminus \Gamma_D)$ we assume

$$\operatorname{mes}\big(\Omega \cap T_R'(x_0)\big) \geq aR^n \tag{6.1}$$

with some $a > 0$, $T_R := B_{2R} \setminus B_R$ and $T_R' := T_R \cap \Omega$.

Dirichlet boundary:

We need for each $x_0 \in \Gamma_D$ that there holds a "*Wiener type condition*".

$$\operatorname{cap}(T_R \cap \Gamma_D; B_{2R}) \geq c_0 R^{n-2} \tag{6.2}$$

For Lipschitz boundary the conditions (6.1) and (6.2) are satisfied. In the case of mixed boundary conditions, one can assume in addition that the set which separates the Dirichlet and Neumann boundary is Lipschitz.

We follow the proof in section 3, however we replace the quantity $\overline{v}_R$ by c_R where

$$c_R = \begin{cases} \overline{v}_R & \text{if } B_{2R} \subset \Omega & (i) \\ 0 & \text{if } B_{2R} \cap \Gamma_D \neq \varnothing & (ii) \\ k_R & \text{if } B_{2R} \cap \partial\Omega \neq \varnothing \text{ and } B_{2R} \cap \Gamma_D = \varnothing & (iii) \end{cases}$$

and k_R is choosen such that

$$\begin{aligned} \operatorname{mes}(\{v \geq k_R\} \cap T_R') &\geq \frac{1}{2}\operatorname{mes}(T_R') \\ \operatorname{mes}(\{v \leq k_R\} \cap T_R') &\geq \frac{1}{2}\operatorname{mes}(T_R') \end{aligned}$$

Note that

$$|k_R| \leq 2 \fint_{T_R'} |v| \,\mathrm{d}x \tag{6.3}$$

We follow the proof of the interior tube-filling condition. In the case that $c_R = \overline{v}_R$ there is no change of the proof.

In case (ii) or (iii) we have a point $x_0' \in \Gamma_D$ or Γ_N such that $x_0' \in \partial\Omega \cap B_{2R}(x_0)$ where

$B_R(x_0) \subset B_{4R}(x_0')$. We take a Lipschitz continuous localization function τ such that

$$\begin{aligned} &\tau = 1 \text{ on } B_{4R}(x_0') \\ &\tau = 0 \text{ on } \mathbb{R}^n \setminus B_{8R}(x_0') \\ &|\nabla \tau| \leq R^{-1} . \end{aligned}$$

Then we replace ζ in (3.2) and (3.3) by τ, and in step (3.4) we replace $\overline{v}_R$ by c_R defined above. We follow the proof of theorem 1 up to (3.4).

In (3.4) the equilibrium of forces is used, so there occurs, in addition the term $\int_{\partial\Omega}(v-c)\zeta\dot{p}do$ which is estimated by

$$K \int\limits_{B_{8R}(x_0')\cap\Omega} |\nabla(v-c)\tau^2| \,\mathrm{d}x \leq K \int\limits_{B_{8R}(x_0')\cap\Omega} |\tau^2 \nabla v| \,\mathrm{d}x + \underbrace{KR^{-1} \int\limits_{B_{8R}(x_0')\cap\Omega} |v-c| \,\mathrm{d}x}_{E_3} . \tag{6.4}$$

The first term on the right hand side in (6.4) is dominated by $KR^{n/2}$ (Young's inequality)

Thus we arrive at an inequality

$$\begin{aligned} c_1 \int_0^T \int\limits_{B_{2R}} \tau^2(|\dot{\sigma}|^2 + |\dot{\xi}|^2) \,\mathrm{d}x \,\mathrm{dt} \leq KR + KR^{-2} \int_0^T \int\limits_{\Omega} |v-c_R|^2 \chi_{B_{8R}(x_0')\setminus B_{4R}(x_0')} \,\mathrm{d}x \,\mathrm{dt} \\ + KR^{-1} \int_0^T \int\limits_{\Omega} |v-c_R| \chi_{B_{8R}(x_0')} \,\mathrm{d}x \,\mathrm{dt} \end{aligned} \tag{6.5}$$

We proceed as in (3.5),(3.6) and (3.8), but with different support for the localization function in order to have the correct setting for the use of Poincaré's inequality.

Due to estimates (3.8),(3.10), the conditions (6.1) and the choice (iii) of k_R and further due to condition (6.1) and the choice of c_R, we may apply Poincaré's inequality as in step (3.6). However with R replaced by $4R$ and T_R' replaced by $T_{4R}'(x_0)$. In the case (ii) the set of zeros of v in $T_R'' := B_{8R}(x_0') \cap \Omega \setminus B_{4R}(x_0')$ is large enough. Hence

$$\int_0^T \int\limits_{T_R''} |v|^2 \,\mathrm{d}x \,\mathrm{d}s \leq KR^2 \int\limits_0^T \int\limits_{T_R''} |\nabla v|^2 \,\mathrm{d}x \,\mathrm{d}s \tag{6.6}$$

In case (iii) with $c_R = k_{4R}$ we estimate

$$\int_0^T \int\limits_{T_R''} |v-k_{4R}|^2 \,\mathrm{d}x \,\mathrm{d}s \leq KR^2 \int_0^T \int\limits_{T_R''} |\nabla v|^2 \,\mathrm{d}x \,\mathrm{d}s \tag{6.7}$$

c.f. section 1.1.3 in [BF02]. We estimate for non negative g

$$\int\limits_{B_{4R}(x_0')} g \, \mathrm{d}x \geq \int\limits_{B_R(x_0)} g \, \mathrm{d}x$$

and

$$\int\limits_{T_R''(x_0')} |\nabla v|^2 \, \mathrm{d}x \leq \int\limits_{B_{16R}(x_0)\setminus B_R(x_0)} |\nabla v|^2 \, \mathrm{d}x \, .$$

Thus we arrive at the analogue of (3.6), but with $T_R = B_{2R} \setminus B_R$ replaced by $B_{16R}(x_0) \setminus B_R(x_0)$ and similarly in (3.11). The additional term E_3 is estimated by

$$\begin{aligned} |E_3| &\leq K \int\limits_{B_{8R}(x_0')\cap\Omega} |\nabla v| \zeta \, \mathrm{d}x \\ &\leq K R^{n/2} \leq K R \end{aligned}$$

using Poincaré's inequality in L^1 .
Here the second part of the proof (3.8)-(3.11) is the same as in section 4, taking into account the different situations where Poincaré's inequality is used.
The tube-filling step thereafter works with concentric balls B_{R_N}, $R_j := R_0 16^{-j}$ and yields a Morrey exponent depending on the geometry of the domain.

A Static hardening

Following Temam [Tem85], there is a stationary model of hardening.

Minimize

$$J(\sigma, \xi) = \frac{1}{2} \int_\Omega (\sigma : (A\sigma) + |\xi|^2) \, \mathrm{d}x$$

with the constraint $F(\sigma, \xi) \leq 0$, where F is the von Mises yield criterion for isotropic and kinematic hardening , over the set of all $(\sigma, \xi) \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}) \times L^2(\Omega, \mathbb{R}^m)$ with

$$\big(\sigma, \nabla v\big) = \langle f, v \rangle + \int_{\Gamma_N} p v do \qquad \forall v \in H^1_{\Gamma_D}(\Omega, \mathbb{R}^n) \, . \tag{A.1}$$

This is, of course the weak form of the balance of forces, where $f \in L^n(\Omega, \mathbb{R}^n)$ is the acting bodyforce and $p \in L^\infty(\partial\Omega, \mathbb{R}^n)$ the surface loading. We further impose the Dirichlet condition

$$u = 0 \text{ on } \Gamma_D \, .$$

on the displacements. This problem can be penalized in the spirit of section 2. We have the penalized problem

Minimize

$$J_\mu(\sigma,\xi) = J(\sigma,\xi) + \frac{1}{2\mu}\int_\Omega (F(\sigma,\xi)_+^2 \,\mathrm{d}x$$

where $(\sigma,\xi) \in L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times L^2(\Omega,\mathbb{R}^m)$ satisfies (A.1).

With similar ideas as exposed in sections 3 and 4, we obtain a Morrey condition

$$\int_{B_R} |\nabla u|^2 \,\mathrm{d}x \le KR^\alpha$$

with some $\alpha > 0$ in n dimensions up to the boundary. Hence the displacement u is Hölder continuous in two space dimensions.

Note that we *only* need L^∞-coefficients of the elastic material tensor A.

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