In this section we prove that a $\varphi_{\overline{a}}$-invariant subspaces are also $\varphi_{\overline{a}}$-hyperinvariants; that is to say, invariant under all linear maps commuting with $\varphi_{\overline{a}}$, (see [8] and [9] for more information about these subspaces).

We need to know the centralizer of $A_{\overline{a}}$. To do that, we first calculate the centralizer of the matrix A_a.

Proposition 2.2 ([7). *] The centralizer $\mathcal{C}(A_a)$ is the set of the matrices X_a in the form:*

$$X_a = \begin{pmatrix} x_n & ax_1 & ax_2 & ax_3 & \dots & ax_{n-2} & ax_{n-1} \\ x_{n-1} & x_n & ax_1 & abx_2 & \dots & ax_{n-3} & ax_{n-2} \\ \vdots & & \ddots & \ddots & & & \\ \vdots & & & \ddots & \ddots & & \\ x_3 & x_4 & x_5 & x_6 & \dots & ax_1 & ax_2 \\ x_2 & x_3 & x_4 & x_5 & \dots & x_n & ax_1 \\ x_1 & x_2 & x_3 & x_4 & \dots & x_{n-1} & x_n \end{pmatrix}$$

Proposition 2.3. *The centralizer $\mathcal{C}(A_{\overline{a}})$ of $A_{\overline{a}}$ is the set of the matrices $Y_{\overline{a}} = SX_aS^{-1}$, if $A_{\overline{a}}S = SA_a$.*

Proof. Proposition 2.2, we have $X_aA_a = A_aX_a$. Then, $SX_aS^{-1}A_{\overline{a}} = A_{\overline{a}}SX_aS^{-1}$. □

Note that if $v = (v_1, \dots, v_n)$ is an eigenvector of $A_{\overline{a}}$, then:

$$\begin{aligned} a_nv_n &= \lambda v_1 \\ a_1v_1 &= \lambda v_2 \\ a_2v_2 &= \lambda v_3 \\ &\vdots \\ a_{n-2}v_{n-2} &= \lambda v_{n-1} \\ a_{n-1}v_{n-1} &= \lambda v_n \end{aligned} \tag{3}$$

In particular, we have that

$$v = \left(\frac{\lambda^{n-1}}{a_1 \dots a_{n-1}}, \frac{\lambda^{n-2}}{a_2 \dots a_{n-1}}, \dots, \frac{\lambda}{a_{n-1}}, 1\right) \tag{4}$$

and the following Proposition holds.

Proposition 2.4. *Let $\lambda \in GF(q)^*$ be an element such that $\lambda^n = \prod_{i=1}^n a_i$. Then, the one-dimensional subspace $[v]$ spanned by the vector v given in (4) is an hyperinvariant subspace.*

Proof.

$$A_{\overline{a}}v = \lambda v$$

and given any $Y_{\overline{a}} \in \mathcal{C}(A_{\overline{a}})$, then

$$\begin{aligned} &Y_{\overline{a}}v = \\ &S(x_nI + x_{n-1}A_{\overline{a}} + x_{n-2}A_{\overline{a}}^2 + \dots + x_1A_{\overline{a}}^{n-1})S^{-1}v \\ &= x_nv + x_{n-1}SA_aS^{-1}v + x_{n-2}SA_a^2S^{-1}v + \dots + \\ &+ x_2SA_a^{n-2}S^{-1}v + x_1SA_{\overline{a}}^{n-1}S^{-1}v \\ &= x_nv + x_{n-1}\lambda v + x_{n-2}\lambda^2 v + \dots + x_1\lambda^{n-1}v \\ &= \alpha v \end{aligned}$$

with $\alpha = x_n + x_{n-1}\lambda + x_2\lambda^2 + \dots + x_2\lambda^{n-2} + x_1\lambda^{n-1} \in \mathbb{F}$. □

Proposition 2.5. *Let F be an invariant subspace of $A_{\overline{a}}$. Then, F is hyperinvariant.*

Proof. It suffices to observe that, for all $Y_{\overline{a}} \in \mathcal{C}(A_{\overline{a}})$,

$$SX_aS^{-1} = x_nI + x_{n-1}A_{\overline{a}} + x_{n-2}A_{\overline{a}}^2 + \dots + x_1A_{\overline{a}}^{n-1}.$$

□