Network Coding Gain of Combination Networks

Chi Kin Ngai
Department of Information Engineering
The Chinese University of Hong Kong
Shatin, New Territories
Hong Kong, China
e-mail: ckngai@alumni.cuhk.net

Raymond W. Yeung
Department of Information Engineering
The Chinese University of Hong Kong
Shatin, New Territories
Hong Kong, China
e-mail: whyeung@ie.cuhk.edu.hk

Abstract — **Network coding theory [1] shows that the multicast rate in a network can be increased if coding is allowed in the network nodes. Thus the capacity of a network with network coding is generally larger than that with routing alone. We quantify this gain in closed form for a class of networks called *combination networks*. ¿From this result, it can readily be deduced that network coding gain can be unbounded.**

I. Introduction

We begin with the definition of a *combination network*[1]. In the sequel, a symbol refers to an element in a given finite field.

Definition 1 *An $\binom{n}{m}$ combination network is a 3-layer single source node multicast network. The first layer consists of the source node, where a message consisting of a number of symbols is generated. The second layer consists of n nodes, where each of them receives a single incoming edge from the source node. The third layer consists of $\binom{n}{m}$ sink nodes, where each of them receives incoming edges from a unique set of m out of the n intermediate nodes on the second layer. The capacity of each edge is equal 1, i.e., one symbol can be transmitted in each edge.*

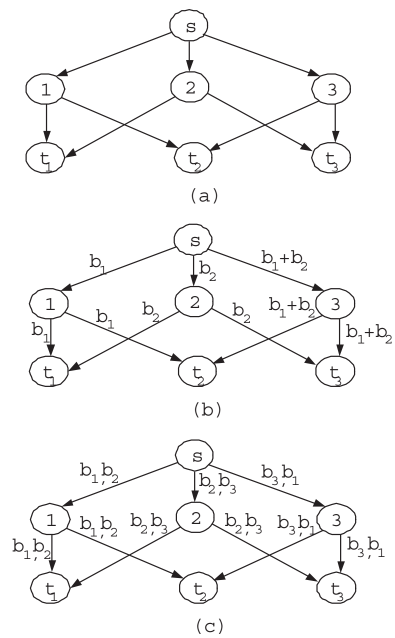

Figure 1: A $\binom{3}{2}$ combination network.

We first explain the notion of *network coding gain* by means of a simple example. Assume all the symbols are binary, and consider the $\binom{3}{2}$ combination network in Fig. 1(a). It is easy to see that

$$\text{maxflow}(s, t_l) = 2 \tag{1}$$

for $l = 1, 2, 3$, so the maximum number of bits that can be multicast from the source node to all the sink nodes is at most 2. This upper bound, called the max-flow bound, can always be achieved by using network coding [1]-[3]. So we define the *network coding capacity* as the max-flow bound. In Fig. 1(b), we show how to achieve the network coding capacity by multicasting 2 bits b_1 and b_2 to all the sink nodes. Note that coding is performed at node 3.

This network is of special interest in practice because it represents a special case of the diversity coding scheme used in commercial disk arrays, which are a kind of fault-tolerant data storage system. Such a system works as follows. Assume the disk array has three disks represented by nodes 1, 2, and 3 in the network, and the information to be stored are the bits b_1 and b_2. The information is encoded into three pieces, namely b_1, b_2 and $b_1 + b_2$, which are stored on the disks represented by nodes 1, 2, and 3, respectively. In the system, there are three decoders, represented by sink nodes t_1, t_2 and t_3, such that each of them has access to a distinct set of two disks. The idea is that when any one disk is out of order, information can still be recovered from the remaining two disks. For example, if the disk represented by node 1 is out of order, then the information can be recovered by the decoder represented by the sink node t_3 which has access to the disks represented by node 2 and node 3. When all the three disks are functioning, the information can be recovered by any decoder.

Let us now consider the case when network coding is not allowed, i.e., only routing can be performed within the network. Let $B = \{b_1, ..., b_\kappa\}$ be the set of bits to be multicast to all the sinks. Let the set of bits sent in the edge s_i be B_i. Since the number of bits being sent must be smaller than the capacity of the edge, we have $|B_i| \leq 1$ for $i = 1, 2, 3$. At node i, the received bits are duplicated and sent in the two out-going edges. Since network coding is not allowed, $B = B_i \cup B_j$ for any $1 \leq i < j \leq 3$. Then we have

$$B_3 \cup (B_1 \cap B_2) = (B_3 \cup B_1) \cap (B_3 \cup B_2) \tag{2}$$
$$= B. \tag{3}$$

Therefore

$$\kappa = |B_3 \cup (B_1 \cap B_2)| \tag{4}$$
$$\leq |B_3| + |B_1 \cap B_2| \tag{5}$$
$$= |B_3| + |B_1| + |B_2| - |B_1 \cup B_2| \tag{6}$$

[1] A combination network is called a uniform bipartite network in [5].

$$= \quad 3-\kappa, \tag{7}$$

which implies $\kappa \leq 1.5$. In Fig. 1(c), we show how 3 bits b_1, b_2, and b_3 can be multicast to all the sinks by sending 2 bits in each edge, i.e., $\kappa = 1.5$ is achieved. Therefore, the *routing capacity* is equal to 1.5, and

$$\text{network coding gain} \;:=\; \frac{\text{network coding capacity}}{\text{routing capacity}} - 1 \;=\; \frac{1}{3}.$$

In this simple example, the network coding gain can easily be determined. Our main contribution in this paper is to determine in the closed form the network coding gain for general combination networks. The rest of the paper is organized as follows. In Section II, we will generalize the result discussed above for $\binom{n}{n-1}$ networks. In Section III, we will further generalize the result for $\binom{n}{m}$ networks. It can then be readily deduced that network coding gain can be unbounded (see also [4]).

II. $\binom{n}{n-1}$ Networks

In this section, we determine the routing capacity of $\binom{n}{n-1}$ combination networks. We will also show that the network coding gain tends to 2 as $n \to \infty$. Since only routing can be performed within the network, without loss of generality, we assume that all the symbols are binary.

Definition 2 *The field F_n generated by sets $B_1, B_2, ..., B_n$ is the collection of sets which can be obtained by any sequence of usual set operations (union, intersection, complement, and difference) on $B_1, B_2, ..., B_n$.*

Definition 3 *The atoms of F_n are sets of the form $\cap_{i=1}^{n} Y_i$ where Y_i is either B_i or ${B_i}^c$, the complement of B_i.*

There are 2^n atoms and 2^{2^n} sets in F_n. Evidently, all the atoms in F_n are disjoint, and each set in F_n can be expressed uniquely as the union of a subset of the atoms of F_n. We assume that the sets $B_1, B_2, ..., B_n$ intersect with each other generically, i.e., all the atoms of F_n are nonempty, unless otherwise specified.

Let A be the set of all atoms, i.e.,

$$A = \{\cap_{i=1}^{n} Y_i | Y_i = B_i \text{ or } {B_i}^c\}. \tag{8}$$

Let

$$A_i = \{(\cap_{j\in K} B_j) \cap (\cap_{i\in\{1,2,\ldots,n\}\setminus K} {B_i}^c) | \; K \subset \{1, ..., n\}, |K| = i\}. \tag{9}$$

In words, A_i is the set of all atoms of F_n which lie in exactly i out of the n sets in $\{B_1, B_2, ..., B_n\}$.

Let X_i be the random variable representing the bits in B_i. We assume that the bits $b_1, b_2, ..., b_\kappa$ are raw bits[2], so that $H(X_i) = |B_i|$. To simplify notation, we will use X_G, where G is any set, to denote $(X_i, i \in G)$. Note that

$$H(X_S) \geq 0, \quad \forall S \in A. \tag{10}$$

[2]Raw bits refer to identical, independent bits, each distributing uniformly on {0,1}.

Lemma 1 $\sum_{k=1}^{n} H(X_{B_1}, ..., X_{B_{k-1}}, X_{B_{k+1}}, ..., X_{B_n} | X_{B_k}) = \sum_{i=1}^{n-1} \{(n-i) \sum_{S_i \in A_i} H(X_{S_i})\}$.

Proof: Let C be any atom in A_i. In other words, $\exists K \subset \{1, 2, ..., n\}$ and $|K| = i$ such that

$$C = \left(\bigcap_{j\in K} B_j\right) \cap \left(\bigcap_{j\in\{1,2,\ldots,n\}\setminus K} {B_j}^c\right). \tag{11}$$

For $1 \leq k \leq n$, let

$$D_k = B_1 \cup ... B_{k-1} \cup B_{k+1} \cup ... \cup B_n \cap {B_k}^c \tag{12}$$

Note that for $S \in A_n$ (i.e., $S = B_1 \cap B_2 \cap \ldots \cap B_n$),

$$S \not\subset D_k, \quad \forall\, 1 \leq k \leq n. \tag{13}$$

Also note that,

$$C \cap D_l = \emptyset, \quad \forall l \in K \tag{14}$$

and

$$C \subset D_l, \quad \forall l \notin K. \tag{15}$$

Therefore,

$$\sum_{k=1}^{n} H(X_{B_1}, ...X_{B_{k-1}}, X_{B_{k+1}}, .., X_{B_n} | X_{B_k})$$

$$= \sum_{k=1}^{n} \{H(X_{B_1\cup\ldots\cup B_n}) - H(X_{B_k})\} \tag{16}$$

$$= \sum_{k=1}^{n} \{H(X_{[(B_1\cup\ldots\cup B_n)\cap {B_k}^c]}) + H(X_{[(B_1\cup\ldots\cup B_n)\cap B_k]}) - H(X_{B_k})\} \tag{17}$$

$$= \sum_{k=1}^{n} \{H(X_{(B_1\cup\ldots\cup B_n)\cap {B_k}^c}) + H(X_{B_k}) - H(X_{B_k})\} \tag{18}$$

$$= \sum_{k=1}^{n} H(X_{(B_1\cap {B_k}^c)\cup\ldots\cup(B_k\cap {B_k}^c)\cup\ldots\cup(B_n\cap {B_k}^c)}) \tag{19}$$

$$= \sum_{k=1}^{n} H(X_{(B_1\cup\ldots B_{k-1}\cup B_{k+1}\cup\ldots\cup B_n)\cap {B_k}^c}) \tag{20}$$

$$= \sum_{k=1}^{n} \sum_{\substack{S\in A\\ S\subset D_k}} H(X_S) \tag{21}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n-1} \sum_{\substack{S\in A_i\\ S\subset D_k}} H(X_S) \tag{22}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n} \sum_{\substack{S\in A_i\\ S\subset D_k}} H(X_S) \tag{23}$$

$$= \sum_{i=1}^{n-1} (n-i) \sum_{S\in A_i} H(X_S), \tag{24}$$

where (17) follows from the assumption that the bits within B_i are raw bits, (22) follows from (13), and (24) follows from (14) and (15) because if $S \in A_i$, then S is a subset of D_k for precisely $(n-i)$ $k's$.

Lemma 2 *For an* $\binom{n}{n-1}$ *combination network, if* κ *is an achievable rate, then*

$$\kappa \leq \frac{n}{2}. \tag{25}$$

Proof: Let

$$H(X_{B_1}, X_{B_2}, ..., X_{B_n}) = \kappa. \tag{26}$$

Since $|B_k| \leq 1$ and hence $H(X_{B_k}) \leq 1$, we have

$$\begin{aligned} & H(X_{B_1}, ..., X_{B_{k-1}}, X_{B_{k+1}}, ..., X_{B_n} | X_{B_k}) \\ &= H(X_{B_1}, X_{B_2}, ..., X_{B_n}) - H(X_{B_k}) \quad (27)\\ &\geq H(X_{B_1}, X_{B_2}, ..., X_{B_n}) - 1 \quad (28)\\ &= \kappa - 1. \quad (29)\end{aligned}$$

Thus

$$\begin{aligned} & \sum_{k=1}^{n} H(X_{B_1}, ..., X_{B_{k-1}}, X_{B_{k+1}}, ..., X_{B_n} | X_{B_k}) \\ &\geq n(\kappa - 1) \quad (30)\\ &= n\kappa - n. \quad (31)\end{aligned}$$

Since the message can be recovered by accessing any of the $n-1$ intermediate nodes, for all $1 \leq k \leq n$, we have

$$\begin{aligned} & H(X_{B_k} | X_{B_1}, ..., X_{B_{k-1}}, X_{B_{k+1}}, ..., X_{B_n}) \quad (32)\\ &= H(X_{B_k \cap B_1{}^c \cap ... \cap B_{k-1}{}^c \cap B_{k+1}{}^c \cap ... \cap B_n{}^c}) \quad (33)\\ &= 0, \quad (34)\end{aligned}$$

or equivalently,

$$H(X_S) = 0, \quad \forall S \in A_1 \tag{35}$$

(cf. (9)). This implies

$$\begin{aligned} & (n-2)H(X_{B_1}, X_{B_2}, ..., X_{B_n}) \\ &= (n-2)\sum_{S \in A} H(X_S) \quad (36)\\ &\geq (n-2)\sum_{i=2}^{n-1}\sum_{S \in A_i} H(X_S) \quad (37)\\ &= \sum_{i=2}^{n-1}(n-2)\sum_{S \in A_i} H(X_S) \quad (38)\\ &\geq \sum_{i=2}^{n-1}(n-i)\sum_{S \in A_i} H(X_S) \quad (39)\\ &= \sum_{i=1}^{n-1}(n-i)\sum_{S \in A_i} H(X_S) \quad (40)\\ &= \sum_{k=1}^{n} H(X_{B_1}, ..., X_{B_{k-1}}, X_{B_{k+1}}, ..., X_{B_n} | X_{B_k}), \quad (41)\end{aligned}$$

where (37) and (39) follow from the non-negativity of entropy, (40) follows from (35), and (41) follows from Lemma 1. Therefore,

$$(n-2)\kappa \geq n\kappa - n \tag{42}$$

so that

$$\kappa \leq \frac{n}{2}, \tag{43}$$

proving the lemma.

Next, we will prove the achievability of the upper bound in Lemma 2.

Lemma 3 *For an* $\binom{n}{n-1}$ *combination network, if*

$$\kappa \leq \frac{n}{2}, \tag{44}$$

then κ *is an achievable rate.*

Proof: We divide the scheme into 2 rounds. During the first round, n independent bits are sent in the n outgoing edges of the source node. During the second round, the n independent bits are shifted to left by one (modulo n) and then sent in the n outgoing edges. For example, during the first round, $b_1, b_2, ..., b_n$ are sent and during the second round, $b_2, ..., b_n, b_1$ are sent. With this scheme, by accessing any $n-1$ intermediate nodes, any sink node can receive n different bits in 2 time units. The upper bound can then be achieved, and the lemma is proved.

By combining the above two lemmas, we obtain

Theorem 1 *For an* $\binom{n}{n-1}$ *combination network,* κ *is an achievable rate if and only if*

$$\kappa \leq \frac{n}{2}. \tag{45}$$

Corollary 1 *For an* $\binom{n}{n-1}$ *combination network, the network coding gain tends to 2 as* $n \to \infty$.

Proof: By Theorem 1, the routing capacity of an $\binom{n}{n-1}$ network is equal to $\frac{n}{2}$ while the network coding capacity (max-flow bound) is equal to $(n-1)$. Therefore,

$$\frac{(n-1)}{\frac{n}{2}} = \frac{2(n-1)}{n} \to 2 \tag{46}$$

as $n \to \infty$.

III. $\binom{n}{m}$ Networks

In this section, we further generalize the results in the last section for $\binom{n}{m}$ networks. We will also show that by choosing n and m appropriately, the network coding gain can become unbounded.

Lemma 4 *For an* $\binom{n}{m}$ *combination network, if* κ *is an achievable rate, then*

$$\kappa \leq \frac{n}{n-m+1}. \tag{47}$$

Proof: Since the message is received by every sink, $\forall K \subset \{1, ..., n\}$ and $|K| \geq m$,

$$\begin{aligned} & H(X_{B_{\{1,2,...,n\}\setminus K}} | X_{B_K}) = 0 & (48) \\ \text{or} \quad & H\left(X_{(\cup_{i\in\{1,2,...,n\}\setminus K} B_i)\cap(\cap_{j\in K} {B_j}^c)}\right) = 0 & (49) \\ \text{or} \quad & H(X_S) = 0 \qquad \forall S \in A_i, 1 \leq i \leq n-m, & (50) \end{aligned}$$

where (50) follows from (9). This implies

$$\begin{aligned} & (m-1)H(X_{B_1}, X_{B_2}, ..., X_{B_n}) \\ = \; & (n-(n-m+1)) \sum_{S\in A} H(X_S) & (51) \\ \geq \; & (n-(n-m+1)) \sum_{i=n-m+1}^{n-1} \sum_{S_i \in A_i} H(X_{S_i}) & (52) \\ = \; & \sum_{i=n-m+1}^{n-1} (n-(n-m+1)) \sum_{S_i \in A_i} H(X_{S_i}) & (53) \\ \geq \; & \sum_{i=n-m+1}^{n-1} (n-i) \sum_{S_i \in A_i} H(X_{S_i}) & (54) \\ = \; & \sum_{i=1}^{n-1} (n-i) \sum_{S_i \in A_i} H(X_{S_i}) & (55) \\ = \; & \sum_{k=1}^{n} H(X_{B_1}, ..., X_{B_{k-1}}, X_{B_{k+1}}, ..., X_{B_n} | X_{B_k}), & (56) \end{aligned}$$

where (52) and (54) follow from the non-negativity of entropy, (55) follows from (50), and (56) follows from Lemma 1. Therefore,

$$(m-1)\kappa \geq n\kappa - n, \qquad (57)$$

or

$$\kappa \leq \frac{n}{n-m+1}, \qquad (58)$$

proving the lemma.

Next, we will prove the achievability of the upper bound in Lemma 4.

Lemma 5 *For an $\binom{n}{m}$ combination network, if*

$$\kappa \leq \frac{n}{n-m+1}, \qquad (59)$$

then κ is an achievable rate.

Proof: Consider an $\binom{n}{m}$ network. Divide the scheme into $n-m+1$ rounds. During the first round, n independent bits are sent on the n outgoing channels of the source node. Then the n bits are shifted to left by one (modulo n) and then sent on the outgoing channels. This step is repeated for the remaining $n-m-1$ rounds. Since the bit sequence received by every sink node is a shifted version of each other and each of them is of length $n-m+1$, they are different from each other. Let $\tilde{B}_k$ be the collection of bits sent in the kth edge from the source node during the $n-m+1$ rounds. Since the $\tilde{B}_k$'s are distinct,

$$\left| \tilde{B}_k \cap \left(\bigcup_{j\neq k} \tilde{B}_j^c \right) \right| \geq 1, \quad \forall\, 1 \leq k \leq n.$$

Hence, for any $1 \leq i_1 < i_2 < ... < i_m \leq n$,

$$\begin{aligned} n \;\geq\; & \left| \bigcup_{j=1}^{m} \tilde{B}_{i_j} \right| & (60) \\ = \; & |\tilde{B}_{i_1}| + \sum_{j=2}^{m} \left| \tilde{B}_{i_j} \cap \left(\bigcap_{l=1}^{j-1} \tilde{B}_{i_l}^c \right) \right| & (61) \\ \geq \; & |\tilde{B}_{i_1}| + \sum_{j=2}^{m} \left| \tilde{B}_{i_j} \cap \left(\bigcap_{l\neq j} \tilde{B}_{i_l}^c \right) \right| & (62) \\ \geq \; & (n-m+1) + (m-1) & (63) \\ = \; & n. & (64) \end{aligned}$$

This implies,

$$\left| \bigcup_{j=1}^{m} \tilde{B}_{i_j} \right| = n. \qquad (65)$$

Therefore, we conclude that each sink node receives all the n bits in $n-m+1$ rounds. The theorem is proved.

By combining the above two lemmas, we obtain

Theorem 2 *For an $\binom{n}{m}$ combination network, κ is an achievable rate if and only if*

$$\kappa \leq \frac{n}{n-m+1}. \qquad (66)$$

Corollary 2 *For an $\binom{n}{m}$ combination network, the network coding gain tends to m as $n \to \infty$.*

Proof: Consider

$$\begin{aligned} \frac{m}{\frac{n}{n-m+1}} \;=\; & \frac{m(n-m+1)}{n} & (67) \\ = \; & m\left(1 - \frac{m}{n} + \frac{1}{n}\right), & (68) \\ & & (69) \end{aligned}$$

which tends to m as $n \to \infty$.

As m can be arbitrarily large, we come to the conclusion that network coding gain can be unbounded. The same conclusion has previously been drawn in [4], where they considered the network coding gain of $\binom{n}{n/2}$ combination networks.

Acknowledgment

The authors thank Prof. Ralf Koetter for pointing out the relation between [4] and the current work. The work described in this paper was partially supported by a grant from the Research Grant Council of the Hong Kong Special Administrative Region, China (RGC Ref. No. CUHK4214/03E).

References

[1] R. Ahlswede, N. Cai, S.-Y. R. Li and R. W. Yeung, "Network Information Flow," *IEEE Trans. Inform. Theory*, IT-46: 1204 -1216, 2000.

[2] S.-Y. R. Li, R. W. Yeung and N. Cai, "Linear network coding," *IEEE Trans. Inform. Theory*, IT-49: 371-381, 2003.

[3] R. Koetter and M. Médard, "An algebraic approach to network coding," *IEEE/ACM Transactions on network coding*, vol. 11, 782-795, 2003.

[4] P. Sanders, S. Egner and L. Tolhuizen, "Polynomial Time Algorithms for Network Information Flow," 15th ACM Symposium on Parallelism in Algorithms and Architectures, Jun. 2003.

[5] Z. P. Li, B. C. Li, D. Jiang and L. C. Lau, "On Achieving Optimal End-to-End Throughput in Data Networks: Theoretical and Empirical Studies," Theoretical and Empirical Studies," ECE Technical Report, University of Toronto, May 2004.