This quotient space construction is usually written as X/A. (Don't confuse this with $X \setminus A$!)

Here's another interesting example. Take the unit disk

$$D^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \leq 1\}.$$

The boundary of the disk is

$$\partial D^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = 1\}.$$

Can you see that $\partial D^2 = S^1$? Now, what is $D^2/\partial D^2$ (or written another way, D^2/S^1)? Imagine taking the boundary of a disk and collapsing it to a single point. Figure 6 shows the process. The result is the 2-sphere S^2:

$$D^2/\partial D^2 = D^2/S^1 \cong S^2.$$

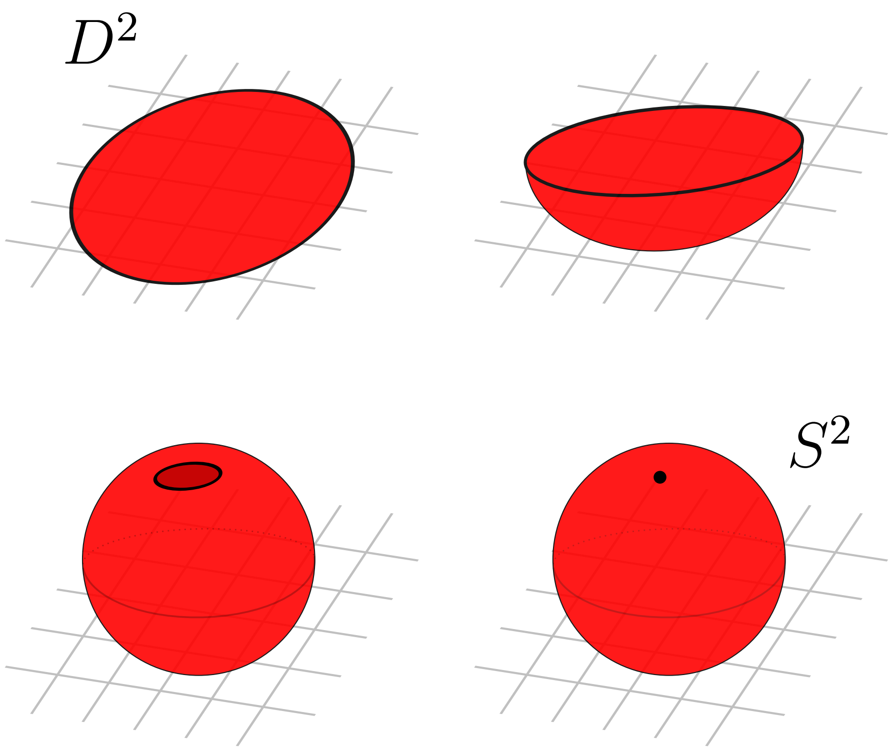

Figure 6: Collapsing the boundary of the disk D^2 to a point, creating S^2.

Nothing in this construction depended on the dimension being two. What is $D^1/\partial D^1$? This can also be written as D^1/S^0.

What is D^1? By definition, it's

$$D^1 = \{x \in \mathbb{R} \mid |x| \leq 1\}.$$

In other words, it's just $[-1, 1]$.

What is S^0, the 0-sphere? Its definition is

$$S^0 = \{x \in \mathbb{R} \mid |x| = 1\}.$$

This is just the set $\{-1, 1\}$. Putting it all together, D^1/S^0 is the interval $[-1, 1]$ with the points -1 and 1 glued together. It's the exact same construction we did earlier to get S^1 (except with a slightly larger interval, but it's topology, so who cares?). Therefore,

$$D^1/\partial D^1 = D^1/S^0 \cong S^1.$$

In any dimension,

$$D^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\},$$

with boundary

$$\partial D^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| = 1\},$$

so that

$$D^n/\partial D^n = D^n/S^{n-1} \cong S^n,$$

but it's hard to picture what this looks like for $n \geq 3$. (For $n = 3$, you can picture the 3-ball D^3 sitting in $\mathbb{R}^3$, but then how would you gather up the boundary sphere and glue it all to a single point? The resulting 3-sphere S^3 lives naturally in four dimensions, which isn't so easy to imagine!)

As another important example, start with $\mathbb{R}^2$ and impose the relation $(x_1, y_1) \sim (x_2, y_2)$ iff both $x_1 - x_2$ and $y_1 - y_2$ are integers. The quotient $\mathbb{R}^2/\sim$ is set equivalent to the product $[0, 1) \times [0, 1)$. (Again, do you see why?) But the topology is not right.

Instead, view this as the unit square $I^2 = I \times I$ with the following relation:

$$(x, y) \sim \begin{cases} (x, y), & 0 < x < 1 \text{ and } 0 < y < 1 \\ (x, 0), & y = 0, 1 \\ (0, y), & x = 0, 1. \end{cases}$$

(At the four corners of the square there is no inconsistency in this definiton: $(0,0) \sim (0,1) \sim (1,0) \sim (1,1)$.) Intuitively this means we should glue the two sides of the square in pairs. Figure 7 shows the result. It's the torus T^2. The torus is only one of many interesting surfaces that can be formed by taking some piece of the plane and gluing various edges together.

If you want to have some fun, see if you can figure out why $T^2 \cong S^1 \times S^1$.

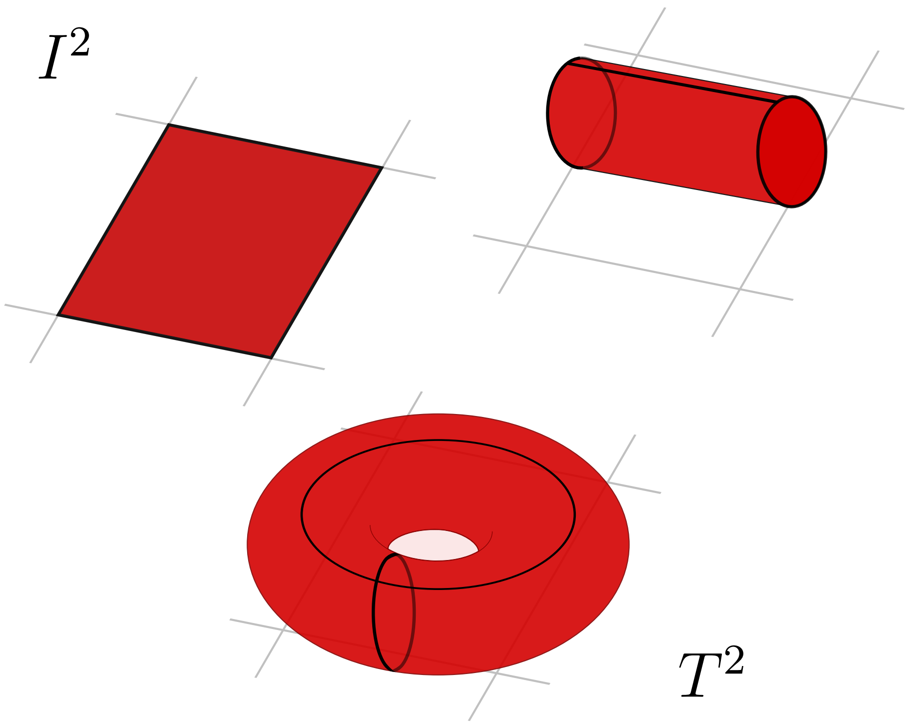

Figure 7: Gluing the edges of a square in pairs gives the torus T^2.

7 and beyond...

Once you have the fundamental four constructions, you can do all sorts of other stuff. Without going into much detail, here's a list of interesting constructions:

- The *wedge sum* $X \vee Y$ is defined by choosing specific points $x \in X$ and $y \in Y$, and then gluing X and Y at a single point, fusing together