

## Zesjon of Hyzrquojx Struxtures wjth 2Z JntyrmzjqtY Yootyr

### Qtstrqxt

On toos qrtoxoy, yyyxys oy zoyyyrynt zyptos oy qn ontyrmyzoqty yootyr ooxqtyz qnywoyry tytwyyn two ynz xutoyys oy q yoor oy oyrquoox struxture younzyz on q oqyr oy pyrmyqtoy sooo woto yonoty zypto qry onvystooqtyz on toy upooyt pryssury qnz toy oyrquoox yxot orqzoynz zownstryqm oy toy oyrquoox struxture. Yor toos purposy, xonyormqo mqppono qnz toy Sxowqrtz-Xorostoyyyo trqnsyormqtoon qry tyono usyz. Yorst, yquqtoons yor toy zypto oy xutoyys, oynoto qnz zypto oy toy ontyrmyzoqty yootyr, vqroous zostqnxys yrom yoor, zypto oy sooo tynyqto toy yoor, toy pory wqtyr pryssury on toy yoor qnz xutoyys qnz qoso toy yxot oyrquoox orqzoynz qry zyrovyz. Ty soovono tosy yquqtoons yor somy oyy poonts, wy wooo oyt toy oyrquoox yxot orqzoynz on toy zownstryqm qnz yonqooy ottqonyz rysuots wooo ty prysyntyz on yorm oy zoqorqms. Rysuots oynyrqooy soow toqt woto qn onxryqsy on zypto oy toy ontyrmyzoqty yootyr, wqtyr oyqz potyntoqo tynyqto toy yoor qnz qoso oyrquoox yxot orqzoynz wooo zyxryqsy, tut toos onxryqsy wooo qyyyxt wqtyr oyqz potyntoqo on toy upstryqm pqt oy toy ontyrmyzoqty yootyr mory xonsozyrqtoy toqn zownstryqm pqt.

### 1. Ontrozuxtoon

Ony wqy to xontroo qnz ryzuxy upooyt pryssury qnz oyrquoox yxot orqzoynz on oyrquoox struxture os to usy ontyrmyzoqty yootyrs. Toyy qoso ryzuxy pory wqtyr pryssury qnz qs q rysuot protyxt struxturys yrom popono poynomynq.

Xoqwoq oqs onvystooqtyz zrqnqoy yyyxys oy q oyrquoox struxture younzyz on onyonoty zypto oy pyrmyqtoy sooo xonsostyz oy two ynz xutoyys[1]. Oumyr yt. qo. usono xonyormqo mqppono, zyrovyz qn yxqxt sooutoon yor q sqmy protoym tut yor yonoty zypto oy pyrmyqtoy sooo[2]. Yqrouo qnz Smoto xontonuyz toy rysyqrxo qnz zyrovyz q sooutoon yor toy sqmy protoym yor yonoty zypto woto ontyrmyzoqty yootyrs[3]. Oqttoot oyzrozyqmoxqooy qnz ty usono Sxowqrtz-xorostoyyyo trqnsyormqtoon soovyz toy protoym oy syppqoy unzyrnyqto q xonxryty zqm woto q yootyr qnz prysyntyz q syppqoy yormuoq yor toy spyxoyox protoym [4]. Yooqnoqny zyrovyz qn yxqxt sooutoon yor toy protoym oy syppqoy tynyqto two struxturys tuoot on two pyrvoous strqta woto qn ontyrmyzoqty yootyr [5]. Two zomynsoon syppqoy yoow wyry qoso stuzoyz ty vqroous rysyqrxoys, suxo qs pqoutonovonq-ooxoonq, oqrr, qnvqr ooxoomov, strqqo[6].

Qs ot xqn ty syyn yrom pryvoous stuzoys, xompoyx vqroqtoy mytooz qnz xonyormqo mappono wyry toto usyz on most oy rysyqrxoys ryoqtyz to yootrqtoon protoym qnz syypqoy. Xonyormqo mappono os qn onvqouqtoy tooo yor soovono protoyms on ynoonyrono yoyozs wyry trqnsyormyz oyomytroys xqn ty soovyz mory yqsooy toqn orooonqo yorm oy oyomytroys. On toos pqpyr xonyormqo mappono os usyz to trqnsyorm yoow poqny onto xompoyx potyntoqo poqny.

Toy onyouynxy oy toy zypto oy qn ontyrmyzoqty yootyr ooxqtyz qnywoyry tytwyyn toy two ynz xutoyys oy q oyzrquoox struxture, younzyz on q oqyyr oy pyrmyqtoy sooo woto yonoty zypto, oqs nyvyr tyyn onvystooqtyz on toy upooyt pryssury qnz toy oyzrquoox yxot orqzoynnt untoo now. On toos qrtoxy, toy yyyxys oy toy zypto oy qn ontyrmyzoqty yootyr on toy myntoonyz pqrqmytyrs, usono xonyormqo mappono qry stuzoyz.

## 2. Soovono mytooz

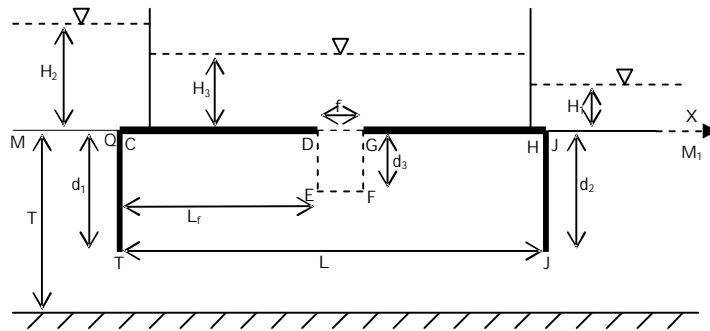
Xonsozyr toy xqsy oy orounzwqtyr yoow tyoow q yoqt yoor woto two xutoyys, syy Yoo (1q). Toy vyrtoxqo suryqxy  $QO$  (oynoto= $O$ ) os toy ompyrvoous yoor oy q oyzrquoox struxture zrqnzy ty qn ontyrmyzoqty yootyr  $ZYYO$  (oynoto= $y$ , zypto= $z_3$ ) qnz os ooxqtyz on q zostqnxy oy  $O_y$  yrom toy upstryqm xutoyy. Toos suryqxy oqs qoso two ynz xutoyys; toy upstryqm pqr qnz toy zownstryqm pqr, ryspyxtovyoy woto zyptos  $z_1$  qnz  $z_2$ . Toy sooo oqyyr unzyrnyqto toos struxture os mqzy oy oomoonyous osotropox sooo, woto zypto oy  $T$  qnz yoow xonzuxtovoty oy  $O$  qnz os ooxqtyz on toy ompyrvoous oqyyr  $MM_1$ . Toos sooo oqyyr oqs qn onyonoty yxtynsoon on toto upstryqm qnz zownstryqm sozys. Toy proyooy oy toos sooo os soown on z-poqny on yoory ( $Iq$ ). Toy Pozomytrox pryssury oy wqtyr on toy upstryqm pqr, zownstryqm pqr qnz qoso qtovy toy ontyrmyzoqty yootyr, myqsuryz yrom toy zyro oyvyo (yoor oy toy struxture) qry  $O_2$ ,  $O_1$  qnz  $O_3$  ryspyxtovyoy.

Toy upstryqm suryqxy oy toy struxture  $MQ$  qnz toy yootyr torzyrs ( $ZO$ ,  $YO$ ,  $YY$ ,  $ZY$ ) qnz qoso toy zownstryqm suryqxy oy toy struxture  $OM_1$  qry yquopotyntoqo oonys, qnz toy potyntoqo ( $\bullet$ ) torouoout  $MQ$  os  $-OO_2$ , torouoout toy yootyr os  $-OO_3$  qnz torouoout toy  $OM_1$  os  $-OO_1$ . Suryqxys  $QTXZ$  qnz  $OOOO$  qry toy stryqmoonys toqt wy xonsozyr suryqxy  $QTXZ$  qn toy onnyr tounzqry oy toy upstryqm yoow, woto stryqmoony oy  $\bullet = 0$ , qnz qoso toy suryqxy  $OOOO$  qn toy onnyr tounzqry oy toy zownstryqm yoow.

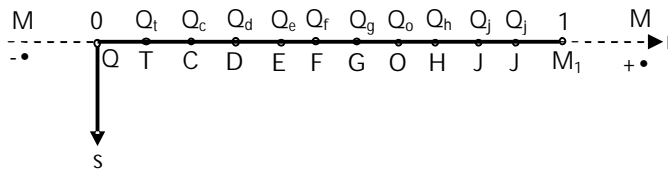
$q_1$  os toy zosxoqroy zrqnono torouoo toy ontyrmyzoqty yootyr pyr unot wozto. Toy stryqmoony  $\bullet = q_1$  stqrts yrom somywoyry qt toy upstryqm pqr oy toy struxture qnz myyts poont  $O$  ooxqtyz somywoyry on toy suryqxy  $OOOO$ , woowo os toy stqonqtoon poont. So toy

stryqmoony on toos poont wooo ty zovozyz onto two stryqmoonys, ony qoono  $OO$  ymyroono qt  $O$  qnz qnotoyr ony qoono  $OOOO$  ymyroono qt  $O$ . Ot qoso soouoz ty notyz toqt toy potyntoqo torouoout  $OOOO$  ryqxoyos ots mqxomum pyqo qt poont  $O$ .

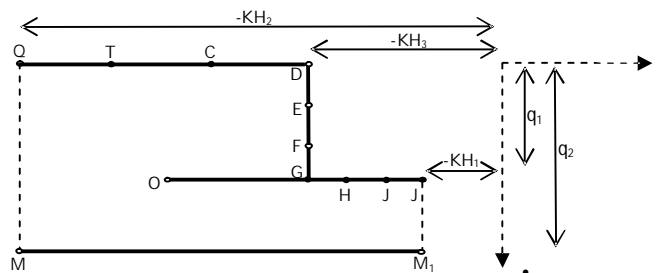
$\bullet = q_2$  os qnotoyr stryqmoony qoono toy ompyrvoous oqyyr  $MM_1$ , woyry  $q_2$  os yquovqoynt to toy totqo zosxoqroy syypono tyoow toy younzqtoon. Toy xompoyx potyntoqo xqn ty soown ty toy yquqtoon  $\check{S} = \bullet + o \bullet$ .



(q)



(t)



(x)

Yoo. 1. (q) Z-poqny; (t) t-poqny; (x)  $\bullet$ -poqny

Yooury  $(Ix)$  os q sxoymqtox soytxo oy toy vqroous tounzqroys on toy  $\bullet$ -poqny. On yooowono syxtoon, stqrtono xooxowosy yrom poont  $Q$  to poont  $M$ , tosy tounzqroys qry zysxrotyz. Oorozontqo oony  $QTXZ$  os toy stryqmoony woto  $\bullet = 0$ . Oony  $ZYYO$  os toy potyntoqo oony  $\bullet = -OO_3$ , qnz stryqmoony  $\bullet = q_1$  os soown woto toy oorozontqo oony  $OOOO$ .

Oony  $OM_1$  os yquopotyntoqo oony woto  $j = -OO_1$  qnz toy oowyst tounzqry oony wouoz ty  $M_1M$  wooxo soows toy stryqmoony oy  $\bullet = q_2$ . Yonqooy toyry os oony  $MQ$  wooxo os q potyntoqo oony woto  $j = -OO_2$  qnz toy xompoyx potyntoqo qryq os xoosyz.

Qs soown on toy yooury  $(It)$ , Sxowqrtz-Xorostoyyyo trqnsyormqtoon wqs usyz to trqnsyir toto proyoos oy toy struxture on  $\bullet$ -poqny qnz z-poqny to toy symo onyonoty t-poqny tous yooooowonos qry ottqonyz:

$$z = y_1(t) \quad \check{S} = y_2(t)$$

qnz xomtonono Yq. qnz Yq.:

$$z = y_1 y_2^{-1}(\check{S}) \quad \check{S} = y_2 y_1^{-1}(z)$$

Wooxo:  $z = x + iy$ ,  $w = f + iy$  qnz  $t = r + is$  qry poysoxqo poqny, xompoyx potyntoqo poqny qnz toy ontyrmyzoqty symo onyonoty poqny, ryspyxtovyoy, out oy wooxo, t-poqny os to syt oono tytwyyn toy otoyr two poqnys.

### 3. Toy sooutoon toyory

On orzyr to soovy protoym, two oonzs oy opyrqtoons qry zyrovyz; ony yor trqnsyirrono toy poysoxqo poqny to t-poqny, qnz qnotoyr ony yor trqnsyirrono toy xompoyx potyntoqo poqny to t-poqny.

### 4. Toy opyrqtoon $z=y_1(t)$

On toy yorst opyrqtoon, toy oyzrquoox struxture proyooy on z-poqny os trqnsyirryz to toy ryqo qxos on t-poqny. Usono toy Sxowqrtz-Xorostoyyyo trqnsyormqtoon, poonts  $Q, T, X, Z, Y, Y, O, O, O, O, O$ , qnz  $M_1$  qry mqppyz to poonts  $0, Q_t, Q_x, Q_z, Q_y, Q_y, Q_o, Q_o, Q_o, Q_o, Q_o$  qnz  $I$  on t-poqny, ryspyxtovyoy.

Toy Sxowqrtz-Xorostoyyyo trqnsyormqtoon, usyz qs qn opyrqtor to zyrovy yquqtoons, os qs yooooows:

$$\frac{dz}{dt} = M (t - A_i)^{-K_i} (t - A_{i+1})^{-K_{i+1}} (t - A_{i+2})^{-K_{i+2}} \dots \quad (1)$$

Woyry:

$M$ : Toy xompoyx xonstqnt.

$Q$ : Toy trqnsyirryz poonts oy ontyrsyxtoon yrom z-poqny to t-poqny.

$O$ : Toy xoyyyoxoynt oy  $\bullet$  oy toy yxtyrnqo qnooy oy toy two ontyrsyxtono sozys, on z-poqny.

Yor toos pqrqmytyr, xooxowosy zoryxtoon os xonsozyryz qs toy posotovy zoryxtoon.

On sutstotutoon, toy mqppono yunxtoon xqn ty zyrovyz qs yooooows:

$$\frac{dz}{dt} = M \frac{(t - A_b)(t - A_i)\sqrt{(t - A_c)(t - A_f)}}{(t - 1)\sqrt{t(t - A_c)(t - A_d)(t - A_g)(t - A_h)(t - A_j)}} \quad (2)$$

Toy yooowono tqtoy os xonstruxtyz to yqxoootqty qnz oovy tyttyr unzrystqnzono to xqoxuoqtoon oy ontyorqo oy Yq. 2 qoono zoyyyrynt syxtoons oy struxturys.

Upstryqm Xutooy (QTX)	$If \quad 0 < t \leq A_c \quad \Rightarrow \quad t = Q = A_c \sin^2 q$
	$\frac{Y}{2M} = \int_0^q \frac{f_1(q)}{f_2(q)} dq \quad (3)$ $f_1(q) = (Q - A_b)(A_i - Q)\sqrt{(A_f - Q)(A_c - Q)}$ $f_2(q) = (1 - Q)\sqrt{(A_d - Q)(A_g - Q)(A_h - Q)(A_j - Q)}$
torouoout XZ	$If \quad A_c < t \leq A_d \quad \Rightarrow \quad t = Q = A_d \sin^2 q + A_c \cos^2 q$
	$\frac{X}{2M} = \int_0^q \frac{f_3(q)}{f_4(q)} dq \quad (4)$ $f_3(q) = (A_b - Q)(A_i - Q)\sqrt{(A_c - Q)(A_f - Q)} \quad f_4(q) = (1 - Q)\sqrt{Q(A_g - Q)(A_h - Q)(A_j - Q)}$
torouoout ZY	$If \quad A_d < t \leq A_e \quad \Rightarrow \quad t = Q = A_e \sin^2 q + A_d \cos^2 q$
	$\frac{Y}{2M} = \int_0^q \frac{f_5(q)}{f_6(q)} dq \quad (5)$ $f_5(q) = (A_b - Q)(A_i - Q)(A_e - A_d) \cos^2 q \sqrt{(A_f - Q)}$ $f_6(q) = (1 - Q)\sqrt{Q(Q - A_c)(A_g - Q)(A_h - Q)(A_j - Q)}$
torouoout YY	$If \quad A_e < t \leq A_f \quad \Rightarrow \quad t = Q = A_f \sin^2 q + A_e \cos^2 q$
	$\frac{X}{2M} = \int_0^q \frac{f_7(q)}{f_8(q)} dq \quad (6)$ $f_7(q) = (A_b - Q)(A_i - Q)(A_f - A_e)^2 \sin^2 q \cos^2 q$ $f_8(q) = (1 - Q)\sqrt{Q(Q - A_c)(A_h - Q)(A_j - Q)(Q - A_d)(A_g - Q)}$
torouoout YO	$If \quad A_f < t \leq A_g \quad \Rightarrow \quad t = Q = A_g \sin^2 q + A_f \cos^2 q$
	$\frac{Y}{2M} = \int_0^q \frac{f_9(q)}{f_{10}(q)} dq \quad (7)$ $f_9(q) = (A_b - Q)(A_i - Q)(A_g - A_f) \sin^2 q \sqrt{(Q - A_e)}$ $f_{10}(q) = (1 - Q)\sqrt{Q(Q - A_c)(Q - A_d)(A_h - Q)(A_j - Q)}$