CLOSED SUBGROUPS OF LIE GROUPS ARE LIE SUBGROUPS

DANIEL LITT

We begin with a relatively easy theorem, which points in the direction of the proof.

Theorem 1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $H \hookrightarrow G$ be a be a subgroup (in the category Grp). Then H is a Lie subgroup if and only if there exists a subspace $V \subset \mathfrak{g}$ and neighborhoods $0 \in U \subset \mathfrak{g}$ and $e \in W \subset G$ such that*

$$\exp|_{U \cap V} : U \cap V \to W \cap H$$

is a diffeomorphism.

Proof. If H is a Lie subgroup, the theorem is obvious. For the converse, note that for $h \in H$, we may consider $\exp|_{U\cap V}^{-1}(h^{-1} \cdot -)$, which is a map $hW \cap H \to U \cap V$. It is clear that these maps cover H and are mutually compatible (as they are diffeomorphisms onto their image in G) and so they induce a manifold structure on H; by construction, this manifold structure is compatible with that of G. □

Theorem 2. *Let G be a Lie group, and $H \hookrightarrow G$ a subgroup (in the category Grp), with closed image. Then H is an embedded Lie subgroup.*

Let $\mathfrak{h}$ be the subset of $\mathfrak{g}$ defined as

$$\mathfrak{h} := \{x \in \mathfrak{g} \mid \exp(tx) \in H, \forall t \in \mathbb{R}\}.$$

That is, the $x \in \mathfrak{g}$ such that the one-parameter subgroup generated by x is entirely contained in H.

Lemma 1. *$\mathfrak{h}$ is a subspace of $\mathfrak{g}$.*

Before proving this we need an auxiliary lemma:

Lemma 2. *Let $x, y \in \mathfrak{g}$. Then*

$$\lim_{n\to\infty} \left(\exp\left(\frac{t}{n} \cdot x\right) \exp\left(\frac{t}{n} \cdot y\right)\right)^n = \exp(t(x+y)).$$

Proof of Lemma 2. Consider the path given by $\gamma : \mathbb{R} \to G$, where $\gamma(t) = \exp(tx)\exp(ty)$. Let $0 \in U \subset \mathfrak{g}$, $e \in W \subset G$ be neighborhoods such that $\exp|_U : U \to W$ is a diffeomorphism, and let $Z(t) = \exp|_U^{-1} \circ \gamma(t)|_{U'}$, where $0 \in U' \subset \mathbb{R}$ is a sufficiently small neighborhood of zero. Then we have

$$\gamma(t) = \exp(Z(t))$$

for $t \in U'$. Note that $dZ|_0 = d\exp^{-1} \circ d(\exp(tx)\exp(ty))|_0 = d(\exp(tx)\exp(ty))|_0 = x + y$ so taking Taylor series, we have

$$\exp(tx)\exp(ty) = \exp(t(x+y) + O(t^2))$$

on some small neighborhood of $0 \in \mathbb{R}$.

But then

$$\begin{aligned}\lim_{n\to\infty} \left(\exp\left(\frac{t}{n} \cdot x\right) \exp\left(\frac{t}{n} \cdot y\right)\right)^n &= \lim_{n\to\infty} \exp(\frac{t}{n} \cdot (x+y) + O((t/n)^2))^n \\ &= \lim_{n\to\infty} \exp(t \cdot (x+y) + O(t^2/n)) \\ &= \exp(t(x+y))\end{aligned}$$

as desired. □

Now we can prove that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$.

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