MEMOIRS
of the
American Mathematical Society

Number 951

Center Manifolds for Semilinear Equations with Non-dense Domain and Applications to Hopf Bifurcation in Age Structured Models

Pierre Magal
Shigui Ruan

November 2009 • Volume 202 • Number 951 (end of volume) • ISSN 0065-9266

American Mathematical Society

MEMOIRS
of the
American Mathematical Society

Number 951

Center Manifolds for Semilinear Equations with Non-dense Domain and Applications to Hopf Bifurcation in Age Structured Models

Pierre Magal
Shigui Ruan

November 2009 • Volume 202 • Number 951 (end of volume) • ISSN 0065-9266

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification*.
Primary 35K90, 35L10, 92D25.

Library of Congress Cataloging-in-Publication Data

Magal, Pierre.

Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models / Pierre Magal, Shigui Ruan.

p. cm. — (Memoirs of the American Mathematical Society, ISSN 0065-9266 ; no. 951)

"Volume 202, number 951 (end of volume)."

Includes bibliographical references.

ISBN 978-0-8218-4653-7 (alk. paper)

1. Evolution equations 2. Cauchy problem. 3. Bifurcation theory 4. Differential equations, Parabolic. I. Ruan, Shigui, 1963- II. Title.

QA377.3.M34 2009
515′.353—dc22 2009029211

Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

Subscription information. The 2009 subscription begins with volume 197 and consists of six mailings, each containing one or more numbers. Subscription prices for 2009 are US$709 list, US$567 institutional member. A late charge of 10% of the subscription price will be imposed on orders received from nonmembers after January 1 of the subscription year. Subscribers outside the United States and India must pay a postage surcharge of US$65; subscribers in India must pay a postage surcharge of US$95. Expedited delivery to destinations in North America US$57; elsewhere US$160. Each number may be ordered separately; *please specify number* when ordering an individual number. For prices and titles of recently released numbers, see the New Publications sections of the *Notices of the American Mathematical Society*.

Back number information. For back issues see the *AMS Catalog of Publications*.

Subscriptions and orders should be addressed to the American Mathematical Society, P. O. Box 845904, Boston, MA 02284-5904 USA. *All orders must be accompanied by payment.* Other correspondence should be addressed to 201 Charles Street, Providence, RI 02904-2294 USA.

Memoirs of the American Mathematical Society (ISSN 0065-9266) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2294 USA. Periodicals postage paid at Providence, RI. Postmaster: Send address changes to Memoirs, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA.

This publication is indexed in *Science Citation Index*®, *SciSearch*®, *Research Alert*®, *CompuMath Citation Index*®, *Current Contents*®/*Physical, Chemical & Earth Sciences*.
Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 14 13 12 11 10 09

Abstract

Several types of differential equations, such as delay differential equations, age-structure models in population dynamics, evolution equations with boundary conditions, can be written as semilinear Cauchy problems with an operator which is not densely defined in its domain. The goal of this paper is to develop a center manifold theory for semilinear Cauchy problems with non-dense domain. Using Liapunov-Perron method and following the techniques of Vanderbauwhede et al. in treating infinite dimensional systems, we study the existence and smoothness of center manifolds for semilinear Cauchy problems with non-dense domain. As an application, we use the center manifold theorem to establish a Hopf bifurcation theorem for age structured models.

Received by the editor June 29, 2007.

Article electronically published on July 22, 2009; S 0065-9266(09)00568-7.

2000 *Mathematics Subject Classification.* Primary 35K90, 37L10; Secondary 92D25.

Key words and phrases. Center manifold, semilinear Cauchy problem, non-dense domain, Hopf bifurcation, age structure model.

Research of the second author was partially supported by NSF grants DMS-0412047, DMS-0715772 and NIH grant R01GM083607.

CHAPTER 1

Introduction

The classical center manifold theory was first established by Pliss [**88**] and Kelley [**65**] and was developed and completed in Carr [**12**], Sijbrand [**95**], Vanderbauwhede [**104**], etc. For the case of a single equilibrium, the center manifold theorem states that if a finite dimensional system has a nonhyperbolic equilibrium, then there exists a center manifold in a neighborhood of the nonhyperbolic equilibrium which is tangent to the generalized eigenspace associated to the corresponding eigenvalues with zero real parts, and the study of the general system near the nonhyperbolic equilibrium reduces to that of an ordinary differential equation restricted on the lower dimensional invariant center manifold. This usually means a considerable reduction of the dimension which leads to simple calculations and a better geometric insight. The center manifold theory has significant applications in studying other problems in dynamical systems, such as bifurcation, stability, perturbation, etc. It has also been used to study various applied problems in biology, engineering, physics, etc. We refer to, for example, Carr [**12**] and Hassard et al. [**52**].

There are two classical methods to prove the existence of center manifolds. The Hadamard (Hadamard [**47**]) method (the graph transformation method) is a geometric approach which bases on the construction of graphs over linearized spaces, see Hirsch et al. [**55**] and Chow et al. [**19, 20**]. The Liapunov-Perron (Liapunov [**71**], Perron [**87**]) method (the variation of constants method) is more analytic in nature, which obtains the manifold as a fixed point of a certain integral equation. The technique originated in Krylov and Bogoliubov [**69**] and was furthered developed by Hale [**48, 49**], see also Ball [**7**], Chow and Lu [**21**], Yi [**112**], etc. The smoothness of center manifolds can be proved by using the contraction mapping in a scale of Banach spaces (Vanderbauwhede and van Gils [**105**]), the Fiber contraction mapping technique (Hirsch et al. [**55**]), the Henry lemma (Henry [**54**], Chow and Lu [**22**]), among other methods (Chow et al. [**18**]). For further results and references on center manifolds, we refer to the monographs of Carr [**12**], Chow and Hale [**16**], Chow et al. [**17**], Sell and You [**94**], Wiggins [**110**], and the survey papers of Bates and Jones [**8**], Vanderbauwhede [**104**] and Vanderbauwhede and Iooss [**106**].

There have been several important extensions of the classical center manifold theory for invariant sets. For higher dimensional invariant sets, it is known that center manifolds exist for an invariant torus with special structure (Chow and Lu [**23**]), for an invariant set consisting of equilibria (Fenichel [**44**]), for some homoclinic orbits (Homburg [**56**], Lin [**72**] and Sandstede [**90**]), for skew-product flows (Chow and Yi [**24**]), for any piece of trajectory of maps (Hirsch et al. [**55**]), and for smooth invariant manifolds and compact invariant sets (Chow et al. [**19, 20**]).

Recently, great attention has been paid to the study of center manifolds in infinite dimensional systems and researchers have developed the center manifold theory for various infinite dimensional systems such as partial differential equations (Bates and Jones [**8**], Da Prato and Lunardi [**30**], Henry [**54**], Scheel [**93**]), semiflows in Banach spaces (Bates et al. [**9**], Chow and Lu [**21**], Gallay [**45**], Scarpellini [**91**], Vanderbauwhede [**103**], Vanderbauwhede and van Gils [**105**]), delay differential equations (Hale [**50**], Hale and Verduyn Lunel [**51**], Diekmann and van Gils [**34, 35**], Diekmann et al. [**36**], Hupkes and Verduyn Lunel [**58**]), infinite dimensional nonautonomous differential equations (Mielke [**81, 82**], Chicone and Latushkin [**15**]), and partial functional differential equations (Lin et al. [**73**], Faria et al. [**43**], Krisztin [**68**], Nguyen and Wu [**83**], Wu [**111**]). Infinite dimensional systems usually do not have some of the nice properties the finite dimensional systems have. For example, the initial value problem may not be well posed, the solutions may not be extended backward, the solutions may not be regular, the domain of operators may not be dense in the state space, etc. Therefore, the center manifold reduction of the infinite dimensional systems plays a very important role in the theory of infinite dimensional systems since it allows us to study ordinary differential equations reduced on the finite dimensional center manifolds. Vanderbauwhede and Iooss [**106**] described some minimal conditions which allow to generalize the approach of Vanderbauwhede [**104**] to infinite dimensional systems.

Let X be a Banach space. Consider the non-homogeneous Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t),\ t \in [0, \tau],\ \ u(0) = x \in \overline{D(A)}, \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator, $f \in L^1((0,\tau), X)$. If $\overline{D(A)} = X$, that is, if $D(A)$ is dense in X, the Cauchy problem has been extensively studied (Kato [**63**], Pazy [**85**]). However, there are many examples (see Da Prato and Sinestrari [**31**]) in which the density condition is not satisfied. Indeed, several types of differential equations, such as delay differential equations, age-structure models in population dynamics, some partial differential equations, evolution equations with nonlinear boundary conditions, can be written as semilinear Cauchy problems with an operator which is not densely defined in its domain (see Thieme [**98, 99**], Ezzinbi and Adimy [**42**], Magal and Ruan [**76**]). Da Prato and Sinestrari [**31**] investigated the existence and uniqueness of solutions to the non-homogeneous Cauchy problem (1.1) when the operator has non-dense domain.

In this paper we present a center manifold theory for semilinear Cauchy problems with non-dense domain. Consider the semiflow generated by the semi-linear Cauchy problem

$$\frac{du}{dt} = Au(t) + F(u(t)),\ t \in [0, \tau],\ \ u(0) = x \in \overline{D(A)},$$

where $F : \overline{D(A)} \to X$ is a continuous map. A very important and useful approach to investigate such non-densely defined problems is to use the integrated semigroup theory, which was first introduced by Arendt [**3, 4**] and further developed by Kellermann and Hieber [**64**], Neubrander [**84**], Thieme [**98, 99**], see also Arendt et al. [**5**] and Magal and Ruan [**76**]. The goal is to show that, combined with the integrated semigroup theory, we can adapt the techniques of Vanderbauwhede [**103, 104**], Vanderbauwhede and Van Gills [**105**] and Vanderbauwhede and Iooss [**106**] to the context of semilinear Cauchy problems with non-dense domain.

As an application, we will apply the center manifold theory for semilinear Cauchy problems with non-dense domain to study Hopf bifurcation in age structure models. Let $u(t,a)$ denote the density of a population at time t with age a. Consider the following age structured model

$$(1.2)\qquad \begin{cases} \dfrac{\partial u(t,a)}{\partial t}+\dfrac{\partial u(t,a)}{\partial a}=-\mu u(t,a),\ a\in(0,+\infty)\,, \\ u(t,0)=\alpha h\left(\int_0^{+\infty}\gamma(a)u(t,a)da\right), \\ u(0,.)=\varphi\in L^1_+\left((0,+\infty)\,;\mathbb{R}\right), \end{cases}$$

where $\mu>0$ is the mortality rate of the population, the function $h(\cdot)$ describes the fertility of the population, $\alpha\geq 0$ is considered as a bifurcation parameter. Such age structured models are hyperbolic partial differential equations (Hadeler and Dietz [**53**], Keyfitz and Keyfitz [**66**]) and have been studied extensively by many researchers since the pioneer work of W. O. Kermack and A. G. McKendrick (Anderson [**1**], Diekmann et al. [**32**], Inaba [**61**]). We refer to some early papers of Gurtin and MacCamy [**46**] and Webb [**107**], the monographs by Hoppensteadt [**57**], Webb [**108**], Iannelli [**59**], and Cushing [**27**], a recent paper of Magal and Ruan [**76**] and the references therein.

The existence of non-trivial periodic solutions in age structured models has been a very interesting and difficult problem, however, there are very few results (Cushing [**25, 26**], Prüss [**89**], Swart [**96**], Kostava and Li [**67**], Bertoni [**10**]). It is believed that such periodic solutions in age structured models are induced by Hopf bifurcation (Castillo-Chavez et al. [**13**], Inaba [**60, 62**], Zhang et al. [**114**]), but there is no general Hopf bifurcation theorem available for age structured models. In this paper we shall use the center manifold theorem for semilinear Cauchy problems with non-dense domain to establish a Hopf bifurcation theorem for the age structured model (1.2).

The paper is organized as follows. In Chapter 2, some results on integrated semigroups are recalled. One of the main tools to develop the center manifold theory is the spectral decomposition of the state space X. The difficulty here is that from the classical theory of C^0-semigroup we only have spectral decomposition of the space $X_0:=\overline{D(A)}$. But in order to deal with non-densely defined problems we need spectral decomposition of the whole state space X. In Chapter 3, we address this issue. In Chapter 4 we present the main results of the paper, namely the existence and smoothness of the center manifold for semilinear Cauchy problems with non-dense domain, by using the Liapunov-Perron method and following the techniques and results of Vanderbauwede and Iooss [**106**].

In Chapter 5, we apply the center manifold theory to study Hopf bifurcation in the age structured model (1.2). This kind of problems has been considered by Diekmann and van Gils [**34, 35**] and Diekmann et al. [**33**] by studying the equivalent integral/delay equations. Nevertheless, here we regard this problem as an example simple enough to illustrate our results. One may observe that the approach used for this kind of problems can be used to study some other types of equations, such as functional differential equations. Once again one of the main difficulties is to obtain the spectral state decomposition for functional differential equations. Notice that this question has been recently addressed for delay differential equations in the space of continuous functions by Liu, Magal and Ruan [**74**] and for neutral delay differential equations in L^p space by Ducrot, Liu and Magal [**39**]. Thus, using these recent developments it is also possible to apply our results presented here to functional differential equations. Of course in the context of functional differential equations this problem was considered in the past (see Hale [**50**]). However, the approach presented here allows us to consider both functional differential equations and age-structured problems as special cases of the non-densely defined problem (Magal and Ruan [**76**]).