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Gravity-wave drag parametrization over complex terrain: The effect of critical-level absorption in directional wind-shear

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SUMMARY

Most schemes to parametrize gravity-wave drag effectively assume that the total wave-stress is associated with a single wave-vector pointing in the direction of the stress vector. However, the wave stress over real terrain is an integral over all azimuthal directions of wave-vector contributions. In unidirectional flow, all of these modes will have a critical line at the same height—if one exists. However, if the wind turns montonically with height, by whatever degree, critical-level absorption *will* occur at all heights for some portion of the wave spectrum. The resultant critical-line wave-drag is always normal to the local wind direction and the wave stress itself turns with height. This contrasts with the height-independent stress assumption used in parametrizing gravity-wave drag outside regions where the wave stress is saturated.

A practical approach to computing the gravity-wave stress over anisotropic orography, which includes this selective critical-level absorption effect, will be presented. This will be supported by an analytical model calculation for constant-shear flow with wind turning.

KEYWORDS: Complex terrain Critical levels Gravity-wave drag Wind shear

1. INTRODUCTION

The linear theory of orographically-forced internal gravity-waves in the atmosphere is at least 50 years old (e.g. Lyra 1940, 1943; Queney 1948) and is the cornerstone of our understanding of mountain-wave dynamics, momentum transport and wave drag. The predicted trapping of gravity-wave energy in the troposphere for commonly occurring airstream profiles of wind and temperature (Scorer 1949) is in accord with our everyday observation of lee-wave clouds in satellite images. Weather forecasters at the Meteorological Office have for many years used a simplified form of the analytical theory of Palm and Foldvik (1960) to forecast the likely occurrence, amplitude and wavelength of trapped lee-waves (see Foldvik 1962; Caswell 1966). Using linear theory, Sawyer (1959) argued that even over moderately hilly terrain, the pressure force exerted by stationary gravity-waves could be comparable with the frictional stress exerted over open countryside typical of southern England. He proposed that gravity-wave drag (GWD) be represented in numerical weather prediction models and suggested a vertical stress-distribution that is linear and decreasing with height to zero at the tropopause. The need to parametrize GWD was further supported by Bretherton (1969) and Lilly (1972), amongst others, but it was not until the work of Boer *et al.* (1984), Palmer *et al.* (1986) and MacFarlane (1987) that it was clearly demonstrated to be a necessary requirement in modern numerical weather prediction and climate modelling.

Another important development in the linear theory of internal gravity-waves was the realization that wave energy could be absorbed at points in the fluid where the intrinsic frequency goes to zero (Bretherton 1966; Booker and Bretherton 1967). Their physical interpretation of the process involved the consideration of wave packets propagating in a flow with sufficiently slow variation in wind and stability (the 'WKB' assumption). It was shown that the group velocity decreases to zero in such a way that a wave packet takes an infinite time to reach a critical line and is effectively absorbed. Booker and Bretherton (1967) clarified the nature of the mathematical singularity in the wave equation at the critical level and showed that almost total absorption of wave energy occurs there provided that the Richardson number there is greater than unity.

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Studies of critical-line absorption have, for the most part, concentrated on unidirectional and horizontally inhomogeneous flows within which the wind speed goes to zero and the flow direction reverses. In an orographic forcing context, this flow would represent the component of the wind directed normal to the axis of two-dimensional orographic ridges. For a general specification of the orography one can consider an integral over all possible wave-vector orientations, and in the special case of unidirectional flows, each wave-vector contribution to the total wave-field has a critical line where the wind speed goes to zero.

Consider for a moment the idealized problem of horizontally-uniform, unidirectional constant-shear flow over sinusoidal orography of infinite extent. If the orography is characterized by a single wave-vector pointing in the same direction as the flow, then the wave stress will similarly remain parallel to the wind. The problem is unchanged by adding an arbitrary mean-flow component normal to the orographic wave-vector and so the wave-stress direction need not bear any particular relation to the local wind direction. However, at the critical level, the wind direction must in general be at right angles to the direction of the wave-drag vector since the component of flow along the wave-vector vanishes there. Notice that in this general (though time-independent) context, critical levels are not associated with zero wind speed. Since the full, linear, orographic-flow problem is an integral over the entire wave-spectrum of two-dimensional flow solutions, one may conclude that *the critical-level drag is always normal to the local wind direction* except for the case where the wind speed goes to zero and the angle is itself ill-defined.

Leaving the mathematical expression of this concept until later, one may go on to deduce that *any* turning of the wind will result in the critical-line absorption of wave-vector contributions normal to the local wind. The accompanying continuous vertical drag-profile is not accounted for in parametrizations of GWD which use a monochromatic idealization of the wave spectrum accompanying the sub-grid-scale orography (i.e. all the momentum flux is attributed to a single wave-vector). Hines (1988) investigated this oversimplification and introduced a bichromatic wave-stress idealization of the sub-grid orography in which two wave-stress vectors separately carry the positive and negative azimuthal contributions to total wave-stress. He also drew attention to the differing critical-level heights of different parts of the wave-number spectrum. In the middle-atmosphere context, Pertsev (1989) has discussed the azimuthal filtering effect of wind turning with height on upward-propagating gravity-waves. Here, wave filtering is caused by trapping of short waves, damping mechanisms (such as radiative relaxation of the temperature field) as well as selective critical-level absorption.

Rather surprisingly, few solutions to the three-dimensional, linear mountain-wave problem have been published which involve wind shear. Sawyer (1962) gave some analytic solutions for multi-layer models for which the wind direction was different in each layer and the waves were trapped: the layer assumption ruled out the possibility of critical levels though. Numerical solutions to the vertical-structure equation (Sawyer 1960, Eq. (10)) are rendered difficult by the presence of critical levels (which are singularities in the differential equation) and resonances. Bretherton (1969) gave some results from a numerical calculation of the vertical gravity-wave momentum-flux over North Wales using real profiles of wind and temperature in a linear model. Although vertical profiles are not given, Bretherton states that the ground-level wave-stress was $0.4\,\mathrm{N\,m^{-2}}$ due west and at a height of 20 km (the model top) it was $0.32\,\mathrm{N\,m^{-2}}$ with a bearing of 262°. This change in magnitude and orientation of the wave stress was due to critical-layer absorption of some wave modes and is precisely the effect that is the main concern of this paper.

Vosper (personal communication, 1994) has developed a computer code to solve the full three-dimensional, steady-state problem and has applied it to study lee waves over the

Lake District (in northern England). Even on the latest supercomputers this calculation is time-consuming, due mainly to the evaluation of the trapped-lee-wave component.

The purpose of this paper is to provide a simplified mathematical framework for describing the vertical flux of horizontal momentum over complex orography due to untrapped gravity-waves. As part of the simplification, the spectrum function of the orographic-height variance is assumed to be separable in total wave-number and azimuth with a power-law dependence for the former. Analytic expressions can then be obtained for the stress variation with height caused by turning of the wind vector. A computed stress-profile is given, based on an analytic solution for constant-shear flow over an isolated circularly symmetric mountain.

2. The momentum-flux integral

The following derivation of the total vertical momentum-flux generated by a rectangular region of hills is adapted from Bretherton (1969) and uses similar nomenclature. Basic-state-density variations are ignored to simplify the analysis: their quantitative effect on the vertical momentum-flux in gravity waves is slight for typical atmospheric flows. Consider an isolated mountainous region within the rectangle defined by $0 \leqslant x \leqslant X$ and $0 \leqslant y \leqslant Y$ with $h(x, y)$, the height of the orography, equal to zero outside this area. Let $\hat{h}(k, l)$ be the double Fourier transform of $h(x, y)$ defined by

$$\hat{h}(k, l) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) \exp\{-\mathrm{i}(kx + ly)\}\, \mathrm{d}x\, \mathrm{d}y \tag{1}$$

with inverse given by

$$h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{h}(k, l) \exp\{\mathrm{i}(kx + ly)\}\, \mathrm{d}k\, \mathrm{d}l \tag{2}$$

with $\hat{h}(k, l) = \hat{h}^*(-k, -l)$ and $(\ )^*$ denotes the complex conjugate. Parseval's equality† can then be used to get an expression for the mean orographic-height variance since

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^2\, \mathrm{d}x\, \mathrm{d}y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{h}|^2 \mathrm{d}k\, \mathrm{d}l \tag{3}$$

or

$$\frac{1}{XY} \int_0^X \int_0^Y h^2\, \mathrm{d}x\, \mathrm{d}y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(k, l) \mathrm{d}k\, \mathrm{d}l \tag{4}$$

where

$$A(k, l) = \frac{4\pi^2}{XY} |\hat{h}|^2 \tag{5}$$

by definition.

Now, if the vertical velocity $w(x, y, z)$ is expressed as a Fourier transform,

$$w(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{w}(k, l, z) \exp\{\mathrm{i}(kx + ly)\}\, \mathrm{d}k\, \mathrm{d}l \tag{6}$$

where $\hat{w}(k, l, z)$ satisfies the vertical-structure equation

$$\frac{\partial^2 \hat{w}}{\partial z^2} + (l_\mathrm{s}^2 - K^2)\hat{w} = 0 \tag{7}$$

† Parseval's *theorem* can be found in most textbooks dealing with Fourier transforms.

in which $l_s(\phi, z)$ is the Scorer parameter given by

$$l_s^2 = \frac{N^2}{U_n^2} - \frac{1}{U_n}\frac{\partial^2 U_n}{\partial z^2}; \tag{8}$$

where N is the buoyancy frequency and $U_n(\phi, z)$ is the mean wind component in the direction of the wave-vector whose magnitude and azimuth angle are given by $K = (k^2 + l^2)^{1/2}$ and ϕ respectively. If $\hat{u}(k, l, z)$ and $\hat{v}(k, l, z)$ are the Fourier transforms of the x and y components of the horizontal-wind perturbation (i.e. $u(x, y, z)$ and $v(x, y, z)$ respectively) then the incompressible form of the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{9}$$

implies that

$$\mathrm{i}(k\hat{u} + l\hat{v}) + \frac{\partial \hat{w}}{\partial z} = 0. \tag{10}$$

The steady-state perturbation momentum equations can be written in quasi-Boussinesq form as

$$U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + wU_z + \frac{\partial}{\partial x}\left(\frac{p'}{\rho_0}\right) = 0 \tag{11}$$

and

$$U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + wV_z + \frac{\partial}{\partial y}\left(\frac{p'}{\rho_0}\right) = 0 \tag{12}$$

where $U(z)$ and $V(z)$ are the basic-state wind components in the x and y directions respectively, p' is the pressure perturbation and $\rho_0(z)$ the basic-state density. Eliminating p' in the above equations and substituting the transform expressions for u and v gives the following relation between $\hat{u}$ and $\hat{v}$

$$\hat{u}l - \hat{v}k = \frac{\mathrm{i}\hat{w}(lU_z - kV_z)}{U_n K}. \tag{13}$$

Using Eq. (10) to eliminate $\hat{v}$ from Eq. (13) gives for $\hat{u}$

$$\hat{u} = \mathrm{i}\frac{\cos\phi}{K}\cdot\frac{\partial \hat{w}}{\partial z} + \frac{\mathrm{i}\hat{w}\sin\phi}{K}\cdot\frac{\sin\phi U_z - \cos\phi V_z}{U_n}. \tag{14}$$

Now, multiplying the transform expression for w by the corresponding expressions for u and then integrating over the entire (x, y) plane gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uw \,\mathrm{d}x\,\mathrm{d}y = 4\pi^2\mathrm{i}\int_0^{\infty}\int_0^{2\pi}\frac{\cos\phi}{K}\frac{\partial \hat{w}}{\partial z}\hat{w}^* K\,\mathrm{d}K\,\mathrm{d}\phi \tag{15}$$

which is equivalent to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uw \,\mathrm{d}x\,\mathrm{d}y = 4\pi^2\mathrm{i}\int_0^{\infty}\int_{-\pi/2}^{\pi/2}\left(\frac{\partial \hat{w}}{\partial z}\hat{w}^* - \frac{\partial \hat{w}^*}{\partial z}\hat{w}\right)\cos\phi\,\mathrm{d}K\,\mathrm{d}\phi. \tag{16}$$

The second term on the right-hand side of Eq. (14) does not contribute to the momentum-flux integral since the resulting quantity is purely imaginary and its azimuthal integral vanishes.

It is helpful to introduce a response function $F(K, \phi)$ given by

$$F(K, \phi) = \frac{1}{2\mathrm{i}|\hat{w}(K, \phi, 0)|^2}\left(\frac{\partial \hat{w}}{\partial z}\hat{w}^* - \frac{\partial \hat{w}^*}{\partial z}\hat{w}\right) \tag{17}$$

where, in accordance with the linearized lower-boundary condition,

$$\hat{w}(K, \phi, 0) = \mathrm{i}KU_0 \cos(\phi - \chi_0)\hat{h}(K, \phi) \tag{18}$$

and where U_0 and χ_0 are the speed and direction, respectively, of the surface wind. Note that $F(K, \phi)$ is independent of height (away from critical lines) for any solution to the vertical-structure equation Eq. (7). Therefore the net surface momentum-flux in the x-direction averaged over the rectangular region of mountains is given by

$$\overline{uw} = -\frac{8\pi^2}{XY}U_0^2 \int_0^\infty \int_{-\pi/2}^{\pi/2} |\hat{h}|^2 F(K, \phi)\cos^2(\phi - \chi_0)\cos\phi K^2\, \mathrm{d}K\, \mathrm{d}\phi \tag{19}$$

and it is easily shown that the vector surface momentum-flux is given by:

$$\overline{\mathbf{V}'w} = -2U_0^2 \int_0^\infty \int_{-\pi/2}^{\pi/2} A(K, \phi)F(K, \phi)\cos^2(\phi - \chi_0)\mathbf{K}K\, \mathrm{d}K\, \mathrm{d}\phi. \tag{20}$$

Now consider the form of $F(K, \phi)$ for hydrostatic waves in a flow with constant wind speed and direction. Equation (7) admits solutions of the form

$$\hat{w} = \mathrm{C}\exp(\mathrm{i}mz) \tag{21}$$

where $m = \pm(l_s^2 - K^2)^{1/2}$ and C is an arbitrary constant. It is then easily shown that $F(K, \phi) = m$. Following Smith (1979) amongst others, we ignore the small error in the momentum-flux integral resulting from the simplification $m = l_s$. Under these conditions Eq. (20) becomes

$$\overline{\mathbf{V}'w} = -2NU_0 \int_0^\infty \int_{-\pi/2}^{\pi/2} A(K, \phi)\cos(\phi - \chi_0)\mathbf{K}K\, \mathrm{d}K\, \mathrm{d}\phi. \tag{22}$$

As an example of the use of Eq. (22) consider the drag exerted on a bell-shaped hill defined by

$$h(r) = \frac{h_0}{(1 + r^2/a^2)^{3/2}} \tag{23}$$

where $r = (x^2 + y^2)^{1/2}$ and a is a characteristic half-width. The Fourier transform of $h(r)$

$$\hat{h}(K) = \frac{h_0 a^2}{2\pi}\exp(-aK) \tag{24}$$

may then be substituted in Eq. (22) and integrated analytically to give

$$\overline{\mathbf{V}'w} = -\frac{N\pi h_0^2 a}{4XY}\mathbf{U}_0 \tag{25}$$

where $\mathbf{U}_0$ is the surface-wind vector. Following Hines (1988), if the normalizing area XY is chosen to be the projected area of the mountain for which $h(r) > h_0/5$ then $XY \approx 1.92401\pi a^2$ and the surface drag is given by

$$\overline{\mathbf{V}'w} \approx -0.129936a^{-1}h_0^2 N\mathbf{U}_0 \tag{26}$$

which agrees with Hines's result.

3. CONSTANT-SHEAR ANALYTIC SOLUTION WITH TURNING

Before proceeding to the detail of the analytic calculation there are some important matters to consider regarding the basic state. Firstly, although the Coriolis force has been ignored in the derivation of the vertical-structure equation, its effect is also implied in the choice of basic state. Vertical wind-shear implies a horizontal temperature-gradient through the thermal-wind relation and turning of the wind with height implies thermal advection. This causes the static stability to be a local function of time if the vertical wind-shear changes with height. One might then consider that this is inconsistent with the search for time-independent wave solutions. One could justifiably assume that the time-scale for these changes in static stability is much longer than the time-scale required to set up the orographic-wave response. For the analytic solution given below this inconsistency does not arise since the shear and implied horizontal temperature-gradient are constant. For a non-rotating system there is no problem with a time-independent basic state in which the wind vector rotates with height.

The simplest type of flow exhibiting wind turning with height is given by $\mathbf{U}_0 = (U_*, \Lambda z)$ where U_* and the shear (Λ) are constant and assumed here to be positive. Wind profiles of this form are characterized by 'warm advection' and are often encountered ahead of developing Atlantic depressions where upper-level north-westerlies overlie south-westerlies associated with an advancing tropical maritime airstream. Although a significant idealization of this synoptic situation, the wind profile used here serves to illustrate the phenomenon of selective critical-level absorption.

The vertical-structure equation for each wave mode (K, ϕ) (Eq. (7)) can be simplified by transforming to a new height coordinate Z defined by

$$Z = K\{z + (U_* \cot\phi)/\Lambda\} \tag{27}$$

so that

$$\frac{\partial^2 \hat{w}}{\partial Z^2} + \left(\frac{Ri(\phi)}{Z^2} - 1\right)\hat{w} = 0 \tag{28}$$

where the Richardson number $Ri(\phi) = N^2/(\Lambda \sin\phi)^2$. Equation (28) has a general solution of the form

$$\hat{w} = Z^{1/2}\left[A I_{\mathrm{i}\mu}(Z) + B I_{-\mathrm{i}\mu}(Z)\right] \tag{29}$$

where $\mu = (Ri - 1/4)^{1/2}$, $I_{\mathrm{i}\mu}(Z)$ is a modified Bessel function of pure imaginary order $\mathrm{i}\mu$ and A and B are constants.

Below the critical level located at $Z = 0$, Z is real and negative (i.e. $Z = |Z| \lim_{\beta\to\pm\pi}\{\exp(\mathrm{i}\beta)\}$) and since

$$I_\nu\{|Z| \exp(\mathrm{i}\pi m)\} = \exp(\mathrm{i}\pi m\nu) I_\nu(|Z|) \tag{30}$$

(Abramowitz and Stegun 1965, Eq. (9.6.30)), where m is an integer, then

$$I_{\mathrm{i}\mu}(Z) = I_{\mathrm{i}\mu}(|Z|) \exp(\pm\pi\mu) \tag{31}$$

for real and negative Z. The appropriate sign in the exponent above is dictated by a causality condition and for negative shear this is negative. Therefore Eq. (29) can be written as

$$\hat{w}(Z) = |Z|^{1/2}\{A I_{\mathrm{i}\mu}(|Z|) \exp(-\pi\mu) + B I_{-\mathrm{i}\mu}(|Z|) \exp(\pi\mu)\}. \tag{32}$$

The reader is referred to Booker and Bretherton (1967) for a full account of the method by which the correct branch is selected. Essentially, they allow wave modes to have a small imaginary phase-speed and consider which side of the real Z axis the critical-line singularity lies.

They also show that for the case of negative shear, the second term in the curly brackets corresponds to upward energy-propagation and the vertical momentum-flux is attenuated by a factor $\exp(-2\mu\pi)$ on passing through the critical level. If the downward energy-propagating mode is rejected by setting $A = 0$ and the $\exp(\mu\pi)$ factor absorbed into B, then

$$\hat{w}(Z) = \begin{cases} B|Z|^{1/2} I_{-\mathrm{i}\mu}(|Z|) & \text{for } Z < 0 \\ B\exp(-\pi\mu)|Z|^{1/2} I_{-\mathrm{i}\mu}(|Z|) & \text{for } Z > 0. \end{cases}$$

Now using Eq. (17) and defining $Z_0 = U_0 K \cot\phi/\Lambda$, the response function $F(K, \phi)$ can be shown to be given by

$$F(K,\phi) = -\frac{K|Z|}{2\mathrm{i}|Z_0||I_{-\mathrm{i}\mu}(|Z_0|)|^2}\left(I_{\mathrm{i}\mu}\frac{dI_{-\mathrm{i}\mu}}{d|Z|} - I_{-\mathrm{i}\mu}\frac{dI_{\mathrm{i}\mu}}{d|Z|}\right) \tag{33}$$

for $Z < 0$, but the term in round brackets is the Wronskian $W(I_{\mathrm{i}\mu}, I_{-\mathrm{i}\mu})$ and has the property that

$$W\left[I_{\mathrm{i}\mu}(|Z|), I_{-\mathrm{i}\mu}(|Z|)\right] = \frac{2\sinh(\pi\mu)}{\mathrm{i}\mu|Z|} \tag{34}$$

(Abramowitz and Stegun 1965, Eq. (9.6.14)) so that Eq. (33) simplifies to

$$F(K,\phi) = \frac{K\sinh(\pi\mu)}{\pi|Z_0||I_{-\mathrm{i}\mu}(Z_0)|^2} \tag{35}$$

for $Z < 0$. Numerical evaluation of F is assisted by writing the Bessel function in the form

$$I_{\mathrm{i}\mu}(z) = \frac{(z/2)^{\mathrm{i}\mu}}{\Gamma(\mathrm{i}\mu)} S(\mu, z) \tag{36}$$

where $S(\mu, z)$ is given by

$$S(\mu, z) = \sum_{k=0}^{k=\infty} \frac{(z^2/4)^k}{k!(\mathrm{i}\mu + k)(\mathrm{i}\mu + k - 1)\ldots\mathrm{i}\mu}. \tag{37}$$

Using the property of gamma functions (Abramowitz and Stegun 1965, Eq. (6.1.29)) that

$$\Gamma(\mathrm{i}\mu)\Gamma(-\mathrm{i}\mu) = \frac{\pi}{\mu\sinh(\pi\mu)} \tag{38}$$

the response function becomes

$$F(K,\phi) = \frac{K}{\mu|Z_0||S(\mu, |Z_0|)|^2} \tag{39}$$

remembering that $Z_0 = Z_0(K, \phi)$ and $\mu = \mu(\phi)$.

Finally, the momentum-flux profile is obtained by substituting Eq. (39) in Eq. (20) with $\chi_0 = 0$ giving

$$\overline{\mathbf{V}'w}(z) = -2U_*^2 \int_0^\infty \int_{-\pi/2}^{\pi/2} \frac{A(K,\phi)H(z_c - z)}{\mu|Z_0||S(\mu, |Z_0|)|^2} \cos^2\phi\,\mathbf{K}\,K^2\,\mathrm{d}K\,\mathrm{d}\phi \tag{40}$$

where $H(\,)$ is the Heaviside function equal to zero(unity) for negative(positive) argument and $z_c = (U_* \cot\phi)/\Lambda$. $H(z_c - z)$ enforces the critical-level absorption effect by switching off the vertical momentum-flux for each wave mode above its critical-level height $z_c(\phi)$. It has been assumed here that the Richardson number at each critical level is greater than unity so that any transmitted momentum-flux can be ignored.

Another way to express this critical-level 'filtering' of wave-momentum contributions is to define a critical azimuth ϕ_c equal to $\chi - \pi/2$ where $\chi(z)$ is the local wind direction. Equation (40) can then be written as

$$\overline{\mathbf{V}'w}(z) = -2U_*^2 \int_0^\infty \int_{\phi_c(z)}^{\pi/2} \frac{A(K,\phi)}{\mu|Z_0||S(\mu,|Z_0|)|^2} \cos^2\phi \mathbf{K} K^2 \,\mathrm{d}K\,\mathrm{d}\phi \qquad (41)$$

and its vertical derivative is given by

$$\frac{\mathrm{d}\overline{\mathbf{V}'w}}{\mathrm{d}z} = 2U_*^2 \int_0^\infty \frac{\mathrm{d}\phi_c}{\mathrm{d}z} \frac{A(K,\phi_c)}{\mu_c|Z_{0c}||S(\mu_c,|Z_{0c}|)|^2} \cos^2\phi_c \mathbf{K} K^2 \,\mathrm{d}K \qquad (42)$$

where $\mu_c = \mu(\phi_c)$ and $Z_{0c} = Z_0(K,\phi_c)$. Furthermore, it can be shown that

$$\frac{\mathrm{d}\phi_c}{\mathrm{d}z} = \frac{\Lambda}{U_*}\sin^2\phi_c \qquad (43)$$

and on substituting for $|Z_{0c}|$ one finds

$$\frac{\mathrm{d}\overline{\mathbf{V}'w}}{\mathrm{d}z} = \frac{2\Lambda^2 \sin^3\phi_c \cos\phi_c}{\mu_c}(\cos\phi_c, \sin\phi_c) \int_0^\infty \frac{A(K,\phi_c)K^2}{|S(\mu_c,|Z_{0c}|)|^2}\,\mathrm{d}K. \qquad (44)$$

Equation (44) has been evaluated for the bell-shaped mountain given by Eq. (23) using its simple transform (Eq. (24)) to provide $A(K,\phi)$. The integral was computed by summing wave contributions at a constant interval of ΔK between $K_{\min}$ and $K_{\max}$. Figure 1 shows a polar plot of the mean drag on unit mass (using the same normalizing area as in Eq. (25)) at 1 km intervals between the surface and a height of 30 km when $U_* = 10\,\mathrm{m\,s^{-1}}$, $\Lambda = 10^{-3}\,\mathrm{s^{-1}}$, $N = 10^{-2}\,\mathrm{s^{-1}}$, the mountain height $h_0 = 600$ m and half-width $a = 4$ km. The annular region in wave-number space over which the integral was evaluated is defined by $K_{\min} = 2\pi/(5\times10^4)\,\mathrm{m^{-1}}$ and $K_{\max} = 2\pi\times10^{-3}\,\mathrm{m^{-1}}$.

Since the wind direction remains between westerly and southerly ($0 < \phi < \pi/2$), the critical-level drag—which is always normal to the local wind—lies between southerly and easterly directions (i.e. $\pi/2 < \phi < \pi$). At low levels the critical-level drag is just east of south and of small amplitude since the corresponding wave-modes have phase lines almost parallel to the surface wind and so are only weakly forced. At about 6 km the magnitude of the drag reaches a maximum and decreases higher up. This can be thought of as due to the increasing height-separation of successive wave-modes suffering critical-level absorption resulting from the monotonic decrease in the rate of change of the wind direction with height.

It is important to emphasise that wave-modes in the first quadrant have been excluded from the above calculation since they have no critical levels. Since all gravity-wave modes in this quadrant are trapped, there exists the possibility of a resonant lee-wave response which would contribute to the net vertical momentum-flux and drag on the orography. The calculation of these modes is beyond the intended scope of this paper.

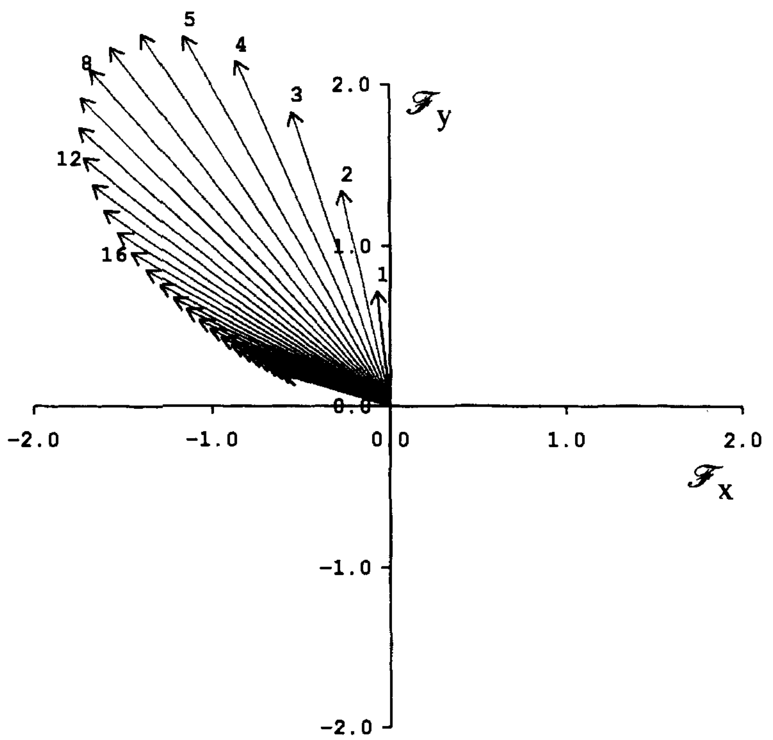

Figure 1. Vectors respresenting the drag $\mathcal{F}$ (on unit mass) caused by critical-level absorption of stationary gravity-waves forced by a bell-shaped hill in an atmosphere where the wind turns monotonically with height. See section 3 of the text for further detail. The units of the components $\mathcal{F}_x$ and $\mathcal{F}_y$ are m s^{-2} $\times$ 10^{-5}.

4. PARAMETRIZATION OF CRITICAL-LEVEL DRAG FOR ARBITRARY WIND-PROFILES

An important practical application of these ideas is in the design of schemes for GWD parametrization (GWDP) for *any* profile of wind and stability. In order to obtain a suitable expression for use in GWDP schemes, the vertical momentum-flux integral (Eq. (20)) will be simplified in two ways. Firstly, it will be assumed that the surface stress is given by Eq. (22) which applies for a uniform airstream under the hydrostatic assumption. This is equivalent to adopting the WKB solution to the vertical-structure equation on the assumption that variations in the mean wind and stability are slow compared to the vertical wavelength of the gravity wave. It implies that there is no partial internal-wave reflection. The second simplification of the momentum-flux integral will involve the choice of a particular separable form for the spectrum function $A(K, \phi)$ of the orographic-height variance. In this way it is possible to deal with mountain anisotropy within the same mathematical framework.

Having established the surface momentum-flux, each wave-vector contribution will be preserved with height until the wind vector becomes normal to that wave vector, at which point critical-level absorption will occur. At any height, therefore, the azimuthal limits of the integral Eq. (22) will change according to the range of wind directions that have been swept out below. The azimuthal sector of wave-vector contributions which are removed by critical-level absorption is at right angles to the sector of wind directions swept out about the surface wind (Fig. 2).

If ϕ_L and ϕ_U are lower and upper azimuths defining the sector in the wave-number

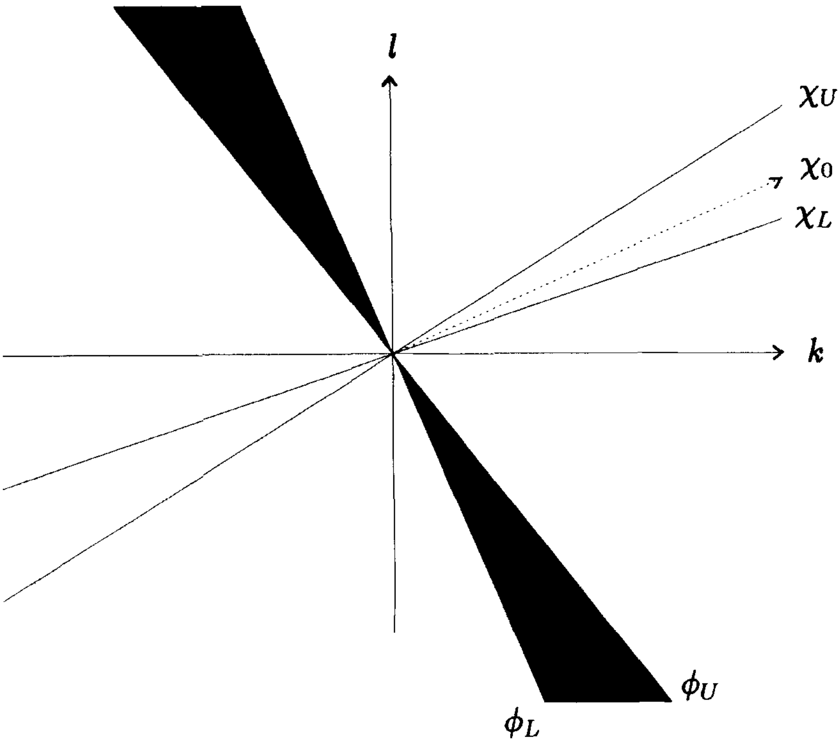

Figure 2. Schematic diagram showing the region of wave-number space excluded from the momentum-flux integral due to critical-level absorption in the atmosphere below. χ_0 represents the surface wind direction and χ_{U} and χ_{L} represent the limits of backing and veering of the wind respectively up to the height in question. k and l are easterly and northerly components of the wave vector; see section 2 of the text.

plane that is removed from the momentum-flux integral by critical-layer absorption, then from Eq. (22) $\tau(z) = \overline{\mathbf{V}'w}$ and

$$\overline{\mathbf{V}'w} = -2NU_0 \int_0^\infty \int_{\phi_{\mathrm{U}}(z)}^{\pi+\phi_{\mathrm{L}}(z)} A(K, \phi)\cos(\phi - \chi_0)K^2(\cos\phi, \sin\phi)\,\mathrm{d}K\,\mathrm{d}\phi. \tag{45}$$

Differentiating the vertical momentum-flux τ with respect to z then gives

$$\frac{\partial \tau}{\partial z} = 2NU_0\left|\frac{\mathrm{d}\phi_*}{\mathrm{d}z}\right|\cos(\phi_* - \chi_0)(\cos\phi_*, \sin\phi_*)\int_0^\infty K^2 A(K, \phi_*)\,\mathrm{d}K \tag{46}$$

where ϕ_* is either ϕ_{L} or ϕ_{U} depending on whether the wind direction is at the lower or upper limit of its range at that height. Note that $\mathrm{d}\phi_*/\mathrm{d}z$ can be expressed as

$$\frac{\mathrm{d}\phi_*}{\mathrm{d}z} = \frac{UV_z - VU_z}{U^2 + V^2} \tag{47}$$

where U and V are the mean wind components evaluated at either ϕ_{L} or ϕ_{U}. As an accuracy test, Eq. (46) can now be used to compute the vertical profile of critical-level wave-drag for the bell-shaped mountain of the previous section and this will be compared to the corresponding exact result from Eq. (44). Using the transform expression for the bell-shaped mountain (Eq. (24)) in the definition of $A(K, \phi)$ and noting that

$$\chi_0 = \phi_{\mathrm{L}} = \frac{\mathrm{d}\phi_{\mathrm{L}}}{\mathrm{d}z} = 0 \tag{48}$$

and

$$\frac{\mathrm{d}\phi_\mathrm{U}}{\mathrm{d}z} = \frac{\Lambda}{U_0} \sin^2 \phi_\mathrm{U} \tag{49}$$

it can be shown that

$$\frac{\partial \tau}{\partial z} = \frac{N \Lambda h_0^2 a}{2XY} \sin^2 \phi_\mathrm{U} \cos \phi_\mathrm{U} (\cos \phi_\mathrm{U}, \sin \phi_\mathrm{U}). \tag{50}$$

Since, by definition, $U_0 \cos \phi_\mathrm{U} + \Lambda z \sin \phi_\mathrm{U} = 0$, ϕ_U may be eliminated from the above expression to give

$$\frac{\partial \tau}{\partial z} = \frac{N \Lambda h_0^2 a}{2XY} \frac{U_0^2 \Lambda z}{(U_0^2 + \Lambda^2 z^2)^2} (\Lambda z, -U_0). \tag{51}$$

As before, the Hines (1988) normalizing area for XY is used and the resulting plots of drag per unit mass with height (for three different values of the the shear Λ) are shown in Fig. 3 together with the corresponding vectors obtained from Eq. (44). The WKB assumption appears to be fairly successful for this range of wind shears and rate of wind turning with height. This suggests that its use in deriving an expression for the parametrized drag-profile is well-founded. Also note from Fig. 3 the increasing magnitude of the drag as the shear increases.

The main problem with the practical application of Eq. (46) in GWDP is the unknown spectrum-function $A(K, \phi)$. Bretherton (1969) proposed a power law of the form γK^b for the scalar spectrum-function $\tilde{A}(K)$ defined by

$$\tilde{A}(K) = \int_0^{2\pi} A(K, \phi) K \, \mathrm{d}\phi. \tag{52}$$

Using gridded terrain-height data for north Wales and performing one-dimensional Fourier transforms he deduced that $b \approx -1.5$. A similar study for the mountains of west Colorado by Young and Pielke (1983) found $b \approx -1$ whilst Bannon and Yuhas (1990) found $b \approx -1.7$ for the Appalachian mountains. These authors deduced the scalar spectrum-function $\tilde{A}$ from one-dimensional (1-D) transforms and then assumed isotropy. Here, a full two-dimensional (2-D) Fourier transform of an orographic-height field has been used and wave-vector contributions to its variance binned into concentric annular regions of the wave-number plane. The width of these annular zones is chosen to be $2\pi/L$ where L is the length of the side of the square domain. The orographic-height data consist of 160×160 points with 1 km separation representing the Lake District region of northern England. Figure 4 shows the binned values of $A(K)$ plotted against wave number on a logarithmic scale. The straight line depicts the roll-off associated with a $K^{-1.5}$ spectrum and is in agreement with Bretherton's finding.

Since a power-law expression for $\tilde{A}$ appears to have some practical utility, the following expression for the spectrum function is proposed

$$KA(K, \phi) = \frac{\gamma K^b}{2\pi} (1 + C_1 \cos 2\phi + C_2 \sin 2\phi) \tag{53}$$

where γ, b, C_1 and C_2 are all treated as constants to be determined. The choice of $\cos 2\phi$ and $\sin 2\phi$ is made so that the condition $A(K, \phi) = A(K, \phi + \pi)$ is automatically satisfied. Clearly the isotropic mountain-range case corresponds to $C_1 = C_2 = 0$. It will be assumed that Eq. (53) applies when $K_\mathrm{L} < K < K_\mathrm{U}$ where K_L and K_U are the smallest and largest wave-numbers resolvable in the orographic-height data. Actually, K_L and K_U should be

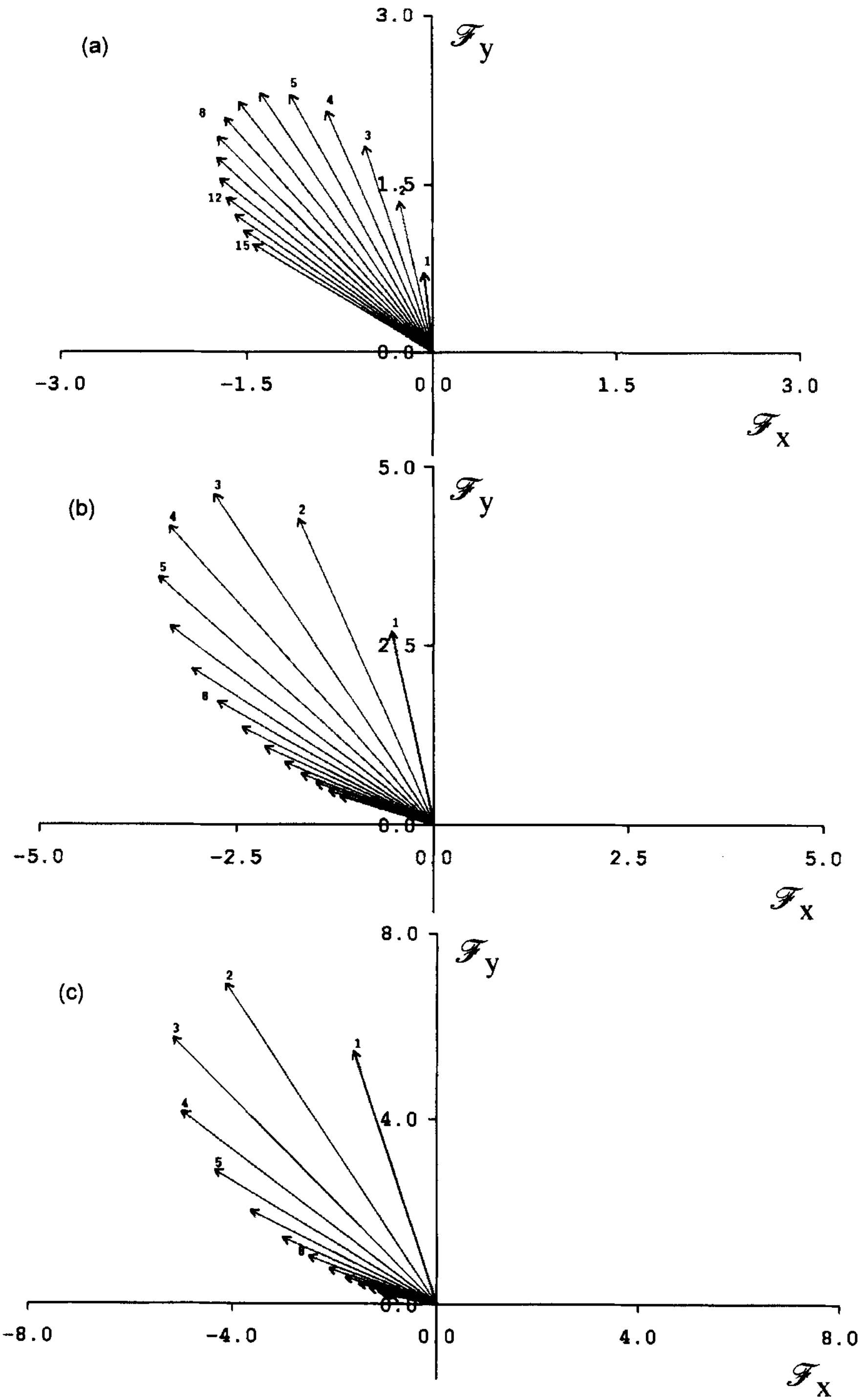

Figure 3. Drag vectors at 1 km intervals for three different values of Λ, the vertical shear of the horizontal wind: (a) and (d) 1×10^{-3} s^{-1}; (b) and (e) 2×10^{-3} s^{-1}; (c) and (f) 3×10^{-3} s^{-1}. (a), (b) and (c) are obtained from the exact analytic formula Eq. (44) whereas (d), (e) and (f) are obtained using the parametrization formula Eq. (46). All other parameters as for Fig. 1.

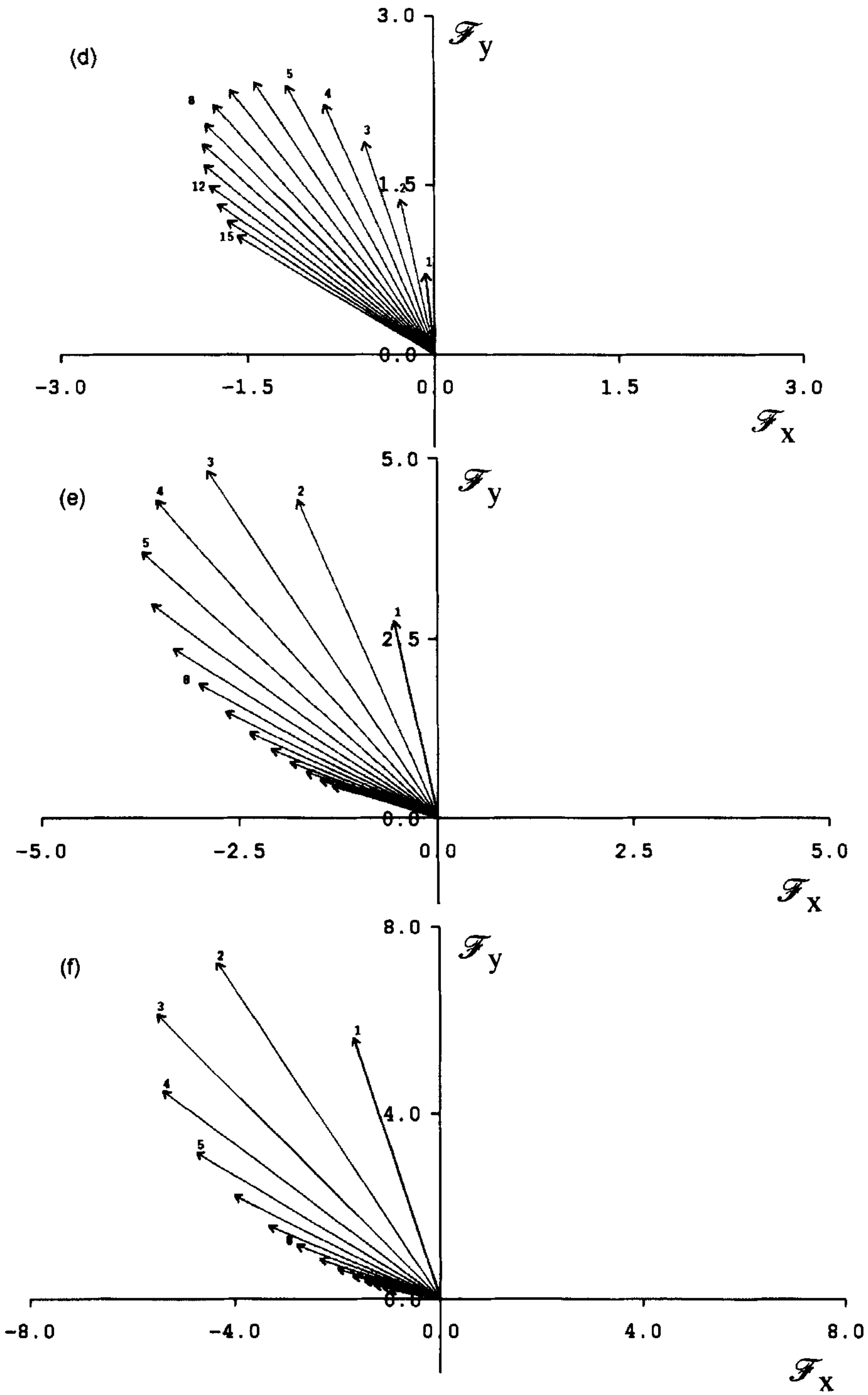

Figure 3. continued.

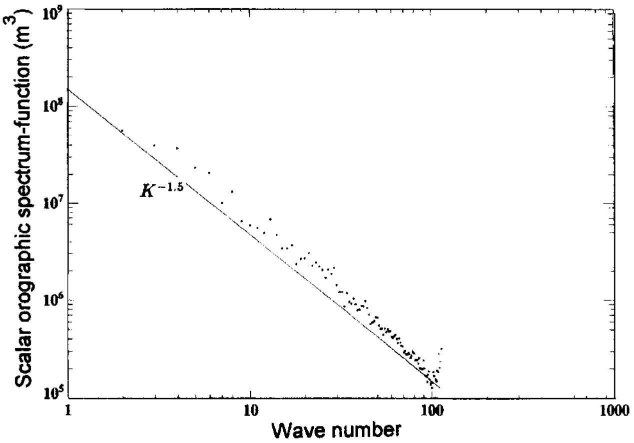

Figure 4. The variation of $A(K)$ with wave number, on logarithmic scales. $A(K)$ is the scalar spectrum-function of the variance of orographic height, obtained by binning spectral contributions from a two-dimensional Fourier series into concentric annular regions of wave-number space. The orographic-height data consists of 160 × 160 points with 1 km separation covering the Lake District of northern England. Wave number is the number of waves in this domain.

limited by $K_U \leqslant N/U_0$ and $K_L \geqslant f/U_0$ where f is the Coriolis parameter. The first of these two requirements excludes scales that are small enough to be outside the hydrostatic regime; the second excludes waves that are long enough to be under inertial control.

The constant C_1 may then be determined from real-height data by multiplying Eq. (53) by cos 2ϕ and then integrating over wave-number space so that

$$C_1 = \frac{b+1}{\gamma(K_U^{b+1} - K_L^{b+1})} \int_{K_L}^{K_U} \int_0^{2\pi} KA(K, \phi) \cos 2\phi \, \mathrm{d}K \, \mathrm{d}\phi \tag{54}$$

and similarly for C_2.

Equation (46) may be evaluated using the power law expression for $A(K, \phi)$ in the wave-number integral between the limits K_L and K_U. Also, the orographic-height variance σ may be written as a wave-number integral so that

$$\sigma = \int_{K_L}^{K_U} \int_0^{2\pi} KA(K, \phi) \, \mathrm{d}K \, \mathrm{d}\phi = \frac{\gamma}{b+1}(K_U^{b+1} - K_L^{b+1}). \tag{55}$$

Equation (46) can then be written as

$$\frac{\partial \tau}{\partial z} = (1 + C_1 \cos 2\phi_* + C_2 \sin 2\phi_*) \cos(\phi_* - \chi_0) \left| \frac{\mathrm{d}\phi_*}{\mathrm{d}z} \right| \hat{K} N U_0 \sigma \hat{\mathbf{n}} \tag{56}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the wind direction (located on the side towards which the wind vector is moving) and

$$\hat{K} = \frac{b+1}{\pi(b+2)}(K_U^{b+2} - K_L^{b+2})(K_U^{b+1} - K_L^{b+1})^{-1}. \tag{57}$$

Consistent with the use of Eq. (53) in the stress equation Eq. (45), the surface stress τ_s is given by

$$\tau_s = \frac{\pi}{4} N U_0 \sigma \hat{K} \left\{ (C_1 + 2) \cos \chi_0 + C_2 \sin \chi_0, \, C_2 \cos \chi_0 - (C_1 - 2) \sin \chi_0 \right\}. \tag{58}$$

Equation (56) (or a similar equation) could be adopted in GWDP schemes in place of the currently used assumption that the sub-saturated wave-stress is independent of height. $\hat{K}$ could be computed from real terrain-height data by estimating b for $K_L < K < K_U$: in practice, though, it would be used as a tuning parameter. The anisotropy parameters C_1 and C_2 would also need to be evaluated from the orographic-height data. Computation of $d\phi_*/dz$ would require storing the upper and lower limits of the range of wind directions sampled whilst moving up from the surface within each grid column (i.e. χ_U and χ_L). The critically absorbed wave-numbers then lie between $\phi = \phi_L = \chi_L - \pi/2$ and $\phi = \phi_U = \chi_U - \pi/2$. It should be noted that $\partial\tau/\partial z$ can only be non-zero if $\phi = \phi_*$.

5. Discussion

Most current GWD schemes employ simplifications which are equivalent to assuming that the wave field is two-dimensional. The wave stress is then assumed to be independent of height until the criterion for wave breaking is satisfied or a critical level is detected. The critical level is deemed to occur when the component of the wind in the direction of the wave stress vanishes. GWD schemes which permit some representation of orographic anisotropy allow the wave stress and surface wind directions to differ (Baines and Palmer 1990) but still treat the stress as a sum of contributions from wave vectors pointing in the same direction.

Hines (1988) used linear theory to examined these weaknesses in GWDP. For mountainous regions that are almost isotropic he proposed representing the wave stress by two azimuthal contributions whose sum is in the direction of the surface wind. Each of these modes carries an equal and opposite contribution normal to the surface wind which would suffer critical-level absorption at different heights. For highly anisotropic mountains he suggested returning to the single-azimuth wave-stress representation where the wave stress lies normal to the orographic ridge axes.

Hines's study is one of few that explicitly recognize the fact that the wave stress is, in general, not height-independent in the linear regime. Only for the special cases of unidirectional flow and/or 2-D orography can the wave stress be assumed to be height-independent away from critical levels (in the sense of the Eliassen–Palm theorem). Under 'real' conditions, changing wind direction with height will remove momentum-flux contributions from some sector of wave-number space and cause the stress vector to rotate and decrease in magnitude. Unlike the classical purely 2-D critical-level problem there is no requirement for the wind speed to go to zero. Of course in the 2-D problem it is always permissible to add a redundant flow-component along the direction of the wave-phase lines without changing the vertical-structure equation. This implies that in general the critical-level drag will be *normal* to the local wind direction. Since the linear 3-D gravity-wave response can be regarded as a Fourier summation of 2-D contributions from all wave-vector directions, this statement will also hold in three dimensions. Its truth can also be shown from a generalization of the Eliassen–Palm approach (Broad, personal communication).

Although the theoretical aspects of the problem discussed here are not new, the physical interpretation of critical-level absorption due to directional wind shear is interesting and some aspects seem unexpected. Unlike the usual 2-D linear view of critical-level absorption where a finite wave-stress is delivered to a zone of infinitesimal thickness, here the drag force predicted by linear theory is finite (e.g. Eq. (46)) and proportional to the angular rate of change of the wind direction. It is conceivable that the linearization assumption has greater validity when the rate of turning of the wind is finite since the gravity wave-action is spread continuously over the layer with directional wind shear rather than delivered to a single height. Presumably, very short vertical wavelengths are realized at small amplitude without wave breaking—unlike the classical 2-D problem where wave breaking always occurs at the critical line. How this might appear to an observer of flow over an isolated hill in a region of wind turning is a matter of some interest.

Observational verification of the effect of wind turning on vertical fluxes of horizontal momentum will not be easy. Computation of momentum fluxes from aircraft data requires extreme care (e.g. taking averages over a whole number of wavelengths after appropriate de-trending of the data) and transience in the wave field causes apparent height variations due to the delay between flight legs in a vertical stack. Mesospheric-Stratospheric-Tropospheric (MST) radar observations are effectively continuous in time but (for a single radar) require the assumption of horizontal homogeneity between the positions of vertical and slanted beam at any level. This may not be valid for short-horizontal-wavelength gravity-waves. Nevertheless, using the Aberystwyth MST radar on the west coast of Wales, Thomas (personal communication) has found vertical profiles of momentum flux which appear to support the critical-level absorption model presented here.

It should be emphasized that the drag force calculated here is only that associated with critical-level effects. The problem of determining the profile of wave stress in wave-breaking regions remains. The expressions derived for $\partial\tau/\partial z$ should be regarded as replacements for the $\partial\tau/\partial z = 0$ assumption used in GWDP outside wave-breaking regions. The net effect of using this more accurate treatment of the linear aspects of GWD should be a more evenly spread vertical distribution of wave stress. This of course depends on how much wind direction variation occurs. There are likely to be characteristic synoptic flow patterns in which directional wind shear is large (e.g. warm/cold advection near fronts). How common these are in a global sense remains to be seen.

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