THE GEOMETRY OF RANK DECOMPOSITIONS OF MATRIX MULTIPLICATION I: 2×2 MATRICES

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Abstract. This is the first in a series of papers on rank decompositions of the matrix multiplication tensor. In this paper we: establish general facts about rank decompositions of tensors, describe potential ways to search for new matrix multiplication decompositions, give a geometric proof of the theorem of [3] establishing the symmetry group of Strassen's algorithm, and present two particularly nice subfamilies in the Strassen family of decompositions.

1. Introduction

This is the first in a planned series of papers on the geometry of rank decompositions of the matrix multiplication tensor $M_{\langle n\rangle}\in \mathbb{C}^{n^2}\otimes\mathbb{C}^{n^2}\otimes\mathbb{C}^{n^2}$. Our goals for the series are to determine possible symmetry groups for potentially optimal (or near optimal) decompositions of the matrix multiplication tensor and eventually to derive new decompositions based on symmetry assumptions. In this paper we study Strassen's rank 7 decomposition of $M_{\langle 2\rangle}$, which we denote Str. In the next paper [1] new decompositions of $M_{\langle 3\rangle}$ are presented and their symmetry groups are described. Although this project began before the papers [3, 4] appeared, we have benefited greatly from them in our study.

We begin in §2 by reviewing Strassen's algorithm as a tensor decomposition. Then in §3 we explain basic facts about rank decompositions of tensors with symmetry, in particular, that the decompositions come in families, and each member of the family has the same abstract symmetry group. While these abstract groups are all the same, for practical purposes (e.g., looking for new decompositions), some realizations are more useful than others. We review the symmetries of the matrix multiplication tensor in §4. After these generalities, in §5 we revisit the Strassen family and display a particularly convenient subfamily. We examine the Strassen family from a projective perspective in §6, which renders much of its symmetry transparent. Generalities on the projective perspective enable a very short proof of the upper bound in Burichenko's determination of the symmetries of Strassen's decomposition [3]. The projective perspective and emphasis on symmetry also enable two geometric proofs that Strassen's expression actually is a decomposition of $M_{\langle 2\rangle}$, which we explain in §7.

Notation and conventions. A,B,C,U,V,W are vector spaces, $GL(A)$ denotes the group of invertible linear maps $A\to A$, and $PGL(A) = GL(A)/\mathbb{C}^*$ the group of projective transformations of projective space $\mathbb{P}A$. If $a\in A$, $[a]$ denotes the corresponding point in projective space. $\mathfrak{S}_d$ denotes the permutation group on d elements. Irreducible representations of $\mathfrak{S}_d$ are indexed by partitions. We let $[\pi]$ denote the irreducible $\mathfrak{S}_d$ module associated to the partition π.

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2. Strassen's algorithm

In 1968, V. Strassen set out to prove the standard algorithm for multiplying $n \times n$ matrices was optimal in the sense that no algorithm using fewer multiplications exists. Since he anticipated this would be difficult to prove, he tried to show it just for two by two matrices. His spectacular failure opened up a whole new area of research: Strassen's algorithm for multiplying 2×2 matrices a, b using seven scalar multiplications [8] is as follows: Set

$$\begin{aligned}
I &= (a^1_1 + a^2_2)(b^1_1 + b^2_2),\\
II &= (a^2_1 + a^2_2)b^1_1,\\
III &= a^1_1(b^1_2 - b^2_2)\\
IV &= a^2_2(-b^1_1 + b^2_1)\\
V &= (a^1_1 + a^1_2)b^2_2\\
VI &= (-a^1_1 + a^2_1)(b^1_1 + b^1_2),\\
VII &= (a^1_2 - a^2_2)(b^2_1 + b^2_2).
\end{aligned}$$

Set

$$\begin{aligned}
c^1_1 &= I + IV - V + VII,\\
c^2_1 &= II + IV,\\
c^1_2 &= III + V,\\
c^2_2 &= I + III - II + VI.
\end{aligned}$$

Then $c = ab$.

To better see symmetry, view matrix multiplication as a trilinear map $(X, Y, Z) \mapsto \operatorname{trace}(XYZ)$ and in tensor form. To view it more invariantly, let $U, V, W = \mathbb{C}^2$, let $A = U^*{\otimes}V$, $B = V^*{\otimes}W$, $C = W^*{\otimes}U$ and consider $M_{\langle 2\rangle} \in (V{\otimes}U^*){\otimes}(W{\otimes}V^*){\otimes}(U{\otimes}W^*)$, where $M_{\langle 2\rangle} = \mathrm{Id}_U \otimes \mathrm{Id}_V \otimes \mathrm{Id}_W$ with the factors re-ordered (see, e.g., [7, §2.5.2]). Write

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ u^1 = (1, 0) \ u^2 = (0, 1) \tag{1}$$

and set $v_j = w_j = u_j$ and $v^j = w^j = u^j$. Then Strassen's algorithm becomes the following tensor decomposition

$$M_{\langle 2\rangle} = (v_1u^1 + v_2u^2){\otimes}(w_1v^1 + w_2v^2){\otimes}(u_1w^1 + u_2w^2) \tag{2}$$

$$\begin{aligned}
&[v_1u^1{\otimes}w_2(v^1 - v^2){\otimes}(u_1 + u_2)w^2 && (3)\\
&+ (v_1 + v_2)u^2{\otimes}w_1v^1{\otimes}u_2(w^1 - w^2)\\
&+ v_2(u^1 - u^2){\otimes}(w_1 + w_2)v^2{\otimes}u_1w^1]\\
&+ [v_2u^2{\otimes}w_1(v^2 - v^1){\otimes}(u_1 + u_2)w^1 && (4)\\
&+ (v_1 + v_2)u^1{\otimes}w_2v^2{\otimes}u_1(w^2 - w^1)\\
&+ v_1(u^2 - u^1){\otimes}(w_1 + w_2)v^1{\otimes}u_2w^2].
\end{aligned}$$

Note that this is the sum of seven rank one tensors, while the standard algorithm in tensor format has eight rank one summands.

Introduce the notation

$$\langle v_i u^j \otimes w_k u^l \otimes u_p w^q \rangle_{Z_3} := v_i u^j \otimes w_k u^l \otimes u_p w^q + v_k u^l \otimes w_p u^q \otimes u_i w^j + v_p u^q \otimes w_i u^j \otimes u_k w^l.$$

Then the decomposition becomes

$$M_{\langle 2 \rangle} = (v_1 u^1 + v_2 u^2) \otimes (w_1 v^1 + w_2 v^2) \otimes (u_1 w^1 + u_2 w^2) \tag{5}$$

$$+ \langle v_1 u^1 \otimes w_2 (v^1 - v^2) \otimes (u_1 + u_2) w^2 \rangle_{Z_3} \tag{6}$$

$$- \langle v_2 u^2 \otimes w_1 (v^1 - v^2) \otimes (u_1 + u_2) w^1 \rangle_{Z_3}. \tag{7}$$

From this presentation we immediately see there is a cyclic Z_3 symmetry by cyclically permuting the factors A, B, C. The Z_3 acting on the rank one elements in the decomposition has three orbits If we exchange $u_1 \leftrightarrow u_2$, $u^1 \leftrightarrow u^2$, $v^1 \leftrightarrow v^2$, etc., the decomposition is also preserved by this Z_2, with orbits (5) and the exchange of the triples, call this an internal Z_2. These symmetries are only part of the picture.

3. Symmetries and families

Let $T \in (C^N)^{\otimes k}$. We say T has rank one if $T = a_1 \otimes \cdots \otimes a_k$ for some $a_j \in C^N$. Define the symmetry group of T, $G_T \subset (GL_N^{\times k}) \ltimes S_k$ to be the subgroup preserving T, where S_k acts by permuting the factors.

For a rank decomposition $T = \sum_{j=1}^r t_j$ with each t_j of tensor rank one, define the set $S := \{t_1, \dots, t_r\}$, which we also call the decomposition, and the symmetry group of the decomposition $\Gamma_S := \{g \in G_T \mid g \cdot S = S\}$. Let $\Gamma_S' = \Gamma_S \cap (GL(A) \times GL(B) \times GL(C))$. Let Str denote Strassen's decomposition of $M_{\langle 2 \rangle}$.

If $g \in G_T$, then $g \cdot S := \{gt_1, \dots, gt_r\}$ is also a rank decomposition of T. Moreover:

Proposition 3.1. For $g \in G_T$, $\Gamma_{g \cdot S} = g \Gamma_S g^{-1}$.

Proof. Let $h \in \Gamma_S$, then $ghg^{-1}(gt_j) = g(ht_j) \in g \cdot S$ so $\Gamma_{g \cdot S} \subseteq g \Gamma_{S_t} g^{-1}$, but the construction is symmetric in $\Gamma_{g \cdot S}$ and Γ_S. ∎

Similarly for a polynomial $P \in S^d C^N$ and a Waring decomposition $P = \ell_1^d + \cdots + \ell_r^d$ for some $\ell_j \in C^N$, and $g \in G_P \subset GL_N$, the same result holds where $S = \{\ell_1, \dots, \ell_r\}$.

In summary, algorithms come in $\dim(G_T)$-dimensional families, and each member of the family has the same abstract symmetry group.

We recall the following theorem of de Groote:

Theorem 3.2. [5] The set of rank seven decompositions of $M_{\langle 2 \rangle}$ is the orbit $G_{M_{\langle 2 \rangle}} \cdot \mathrm{Str}$.

4. Symmetries of $M_{\langle n \rangle}$

We review the symmetry group of the matrix multiplication tensor

$$G_{M_{\langle n \rangle}} := \{g \in GL_{n^2}^{\times 3} \times S_3 \mid g \cdot M_{\langle n \rangle} = M_{\langle n \rangle}\}.$$

One may also consider matrix multiplication as a polynomial that happens to be multi-linear, $M_{\langle n \rangle} \in S^3(A \oplus B \oplus C)$, and consider

$$\tilde{G}_{M_{\langle n \rangle}} := \{g \in GL(A \oplus B \oplus C) \mid g \cdot M_{\langle n \rangle} = M_{\langle n \rangle}\}.$$

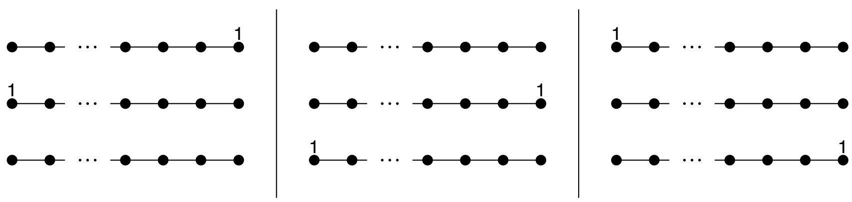
Note that $(GL(A) \times GL(B) \times GL(C)) \times S_3 \subset GL(A \oplus B \oplus C)$, so $G_{M_{\langle n \rangle}} \subseteq \tilde{G}_{M_{\langle n \rangle}}$.

It is clear that $PGL_n \times PGL_n \times PGL_n \times \mathbb{Z}_3 \subset G_{M_{\langle n \rangle}}$, the $\mathbb{Z}_3$ because $\mathrm{trace}(XYZ) = \mathrm{trace}(YZX)$, and the PGL_n's appear instead of GL_n because if we rescale by $\lambda \mathrm{Id}_U$, then U^* scales by $\frac{1}{\lambda}$ and there is no effect on the decomposition. Moreover since $\mathrm{trace}(XYZ) = \mathrm{trace}(Y^T X^T Z^T)$, we have $PGL_n^{\times 3} \ltimes D_3 \subseteq G_{M_{\langle n \rangle}}$, where the dihedral group D_3 is isomorphic to S_3, but we denote it by D_3 to avoid confusion with a second copy of S_3 that will appear. We emphasize that this $\mathbb{Z}_2$ is not contained in either the S_3 permuting the factors or the $PGL(A) \times PGL(B) \times PGL(C)$ acting on them. In $\tilde{G}_{M_{\langle n \rangle}}$ we can also rescale the three factors by non-zero complex numbers λ, μ, ν such that $\lambda\mu\nu = 1$, so we have $(\mathbb{C}^*)^{\times 2} \times PGL_n^{\times 3} \ltimes D_3 \subseteq \tilde{G}_{M_{\langle n \rangle}}$,

We will be primarily interested in $G_{M_{\langle n \rangle}}$. The first equality in the following proposition appeared in [5, Thms. 3.3,3.4] and [4, Prop. 4.7] with ad-hoc proofs. The second assertion appeared in [6]. We reproduce the proof from [6], as it is a special case of the result there.

Proposition 4.1. $G_{M_{\langle n \rangle}} = PGL_n^{\times 3} \ltimes D_3$ and $\tilde{G}_{M_{\langle n \rangle}} = (\mathbb{C}^*)^{\times 2} \times PGL_n^{\times 3} \ltimes D_3$.

Proof. It will be sufficient to show the second equality because the $(\mathbb{C}^*)^{\times 2}$ acts trivially on $A \otimes B \otimes C$. For polynomials, we use the method of [2, Prop. 2.2] adapted to reducible representations. A straight-forward Lie algebra calculation shows the connected component of the identity of $\tilde{G}_{M_{\langle n \rangle}}$ is $\tilde{G}^0_{M_{\langle n \rangle}} = (\mathbb{C}^*)^{\times 2} \times PGL_n^{\times 3}$. As was observed in [2] the full stabilizer group must be contained in its normalizer $N(\tilde{G}^0_{M_{\langle n \rangle}})$. But the normalizer is the automorphism group of the marked Dynkin diagram for $A \oplus B \oplus C$, which in our case is



There are three triples of marked diagrams. Call each column consisting of 3 marked diagrams a group. The automorphism group of the picture is $D_3 = \mathbb{Z}_2 \ltimes \mathbb{Z}_3$, where the $\mathbb{Z}_2$ may be seen as flipping each diagram, exchanging the first and third diagram in each group, and exchanging the first and second group. The $\mathbb{Z}_3$ comes from cyclically permuting each group and the diagrams within each group. $\square$

Regarding the symmetries discussed in §2, the $\mathbb{Z}_3$ is in the S_3 in $PGL_2^{\times 3} \times S_3$ and the internal $\mathbb{Z}_2$ is in $\Gamma'_{Str} \subset PGL_2^{\times 3}$.

Thus if S is (the set of points of) a rank decomposition of $M_{\langle n \rangle}$, then $\Gamma_S \subset [(GL(U) \times GL(V) \times GL(W)) \ltimes \mathbb{Z}_3] \ltimes \mathbb{Z}_2$.

We call a $\mathbb{Z}_3 \subset \Gamma_S$ a standard cyclic symmetry if it corresponds to $(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}) \cdot \mathbb{Z}_3 \subset (GL(U) \times GL(V) \times GL(W)) \ltimes \mathbb{Z}_3$.

We call a $\mathbb{Z}_2 \subset \Gamma_S$ a convenient transpose symmetry if it corresponds to the symmetry of $M_{\langle n \rangle}$ given by $a \otimes b \otimes c \mapsto a^T \otimes c^T \otimes b^T$. The convenient transpose symmetry lies in $(GL(A) \times GL(B) \times GL(C)) \times S_2 \subset (GL(A) \times GL(B) \times GL(C)) \times S_3$, where the component of the transpose in S_2 switches the last two factors and the component in $GL(A) \times GL(B) \times GL(C)$ sends each matrix to its transpose.

Remark 4.2. Since $M_{\langle n \rangle} \in (U^* \otimes U)^{\otimes 3}$ one could consider the larger symmetry group considering $M_{\langle n \rangle} \in U^{\otimes 3} \otimes U^{*\otimes 3}$ as is done in [3].

5. The Strassen family

Since $PGL_2^{\times 3} \subset G_{M_{\langle 2\rangle}}$, we can replace u_1, u_2 by any basis of U in Strassen's decomposition, and similarly for v_1, v_2 and w_1, w_2. In particular, we need not have $u_1 = v_1$ etc... When we do that, the symmetries become conjugated by our change of basis matrices. If we only use elements of the diagonal PGL_2 in $PGL_2^{\times 3}$, the Z_3-symmetry remains standard. More subtly, the Z_3-symmetry remains the standard cyclic permutation of factors if we apply elements of Z_3 in any of the PGL_2's, i.e., setting $\omega = e^{\frac{2\pi i}{3}}$, we can apply any of

$$\rho(\omega) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \rho(\omega^2) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

to U, V or W.

For example, if we apply the change of basis matrices

$$g_U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in GL(U), \quad g_V = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \in GL(V), \quad g_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(W),$$

and take the image vectors as our new basis vectors, then setting $u_3 = -(u_1 + u_2)$ and $u^3 = u^1 - u^2$ and similarly for the v's and w's, the decomposition becomes:

$$M_{\langle 2\rangle} = -(v_1u^2 + v_2u^3) \otimes (w_1v^2 + w_2v^3) \otimes (u_1w^2 + u_2w^3) \tag{8}$$
$$+ v_1u^1 \otimes w_1v^1 \otimes u_1w^1 \tag{9}$$
$$+ v_3u^2 \otimes w_3v^2 \otimes u_3w^2 \tag{10}$$
$$+ v_2u^3 \otimes w_2v^3 \otimes u_2w^3 \tag{11}$$
$$- \langle v_1u^2 \otimes w_2v^1 \otimes u_3w^3 \rangle_{Z_3}. \tag{12}$$

Remark 5.1. The matrices in (12) are all nilpotent, and none of the other matrices appearing in this decomposition are.

Notice that for the first term $v_1u^2 + v_2u^3 = v_2u^1 + v_3u^1 = v_3u^2 + v_2u^1$. Here there is a standard $Z_3 \subset S_3$. There are four fixed points for this standard Z_3: (8),(9)(10),(11). (In any element of the Strassen family there will be some Z_3 with four fixed points, but the Z_3 need not be standard.) There is also a standard $Z_3 \subset PGL_2^{\times 3}$ embedded diagonally, that sends $u_1 \to u_2 \to u_3$, and acting by the inverse matrix on the dual basis, and similarly for the v's and w's. Under this action (8) is fixed and we have the cyclic permutation (9)→(11)→(10).

If we take the standard vectors of (1) in each factor we get

$$M_{\langle 2\rangle} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^{\otimes 3} + \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}^{\otimes 3} + \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle_{Z_3}$$

If we want to see the $Z_3 \subset PGL_2^{\times 3}$ more transparently, it is better to diagonalize the Z_3 action so the first matrix becomes

$$a = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

where $\omega := \exp(\frac{-2\pi i}{3})$. Then for $\iota := i/\sqrt{3}$, $\sigma := \exp(\frac{2\pi i}{12})/\sqrt{3}$ we get

$$M_{\langle 2\rangle} = a^{\otimes 3} + b^{\otimes 3} + (\square(b))^{\otimes 3} + (\square^2(b))^{\otimes 3} + \langle c \otimes \square(c) \otimes \square^2(c) \rangle_{Z_3},$$

where

$$b := \begin{pmatrix} \sigma & \bar{\iota} \\ \iota & \bar{\sigma} \end{pmatrix}, \quad c := \begin{pmatrix} \iota & \iota \\ \bar{\iota} & \bar{\iota} \end{pmatrix}, \quad \square : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}, \square(X) = aXa^{-1}.$$

Note that $a + b + c = 0$.

6. Projective perspective

Although the above description of the Strassen family of decompositions for $M_{\langle 2\rangle}$ is satisfying, it becomes even more transparent with a projective perspective.

6.1. $M_{\langle 2\rangle}$ *viewed projectively.* Recall that PGL_2 acts simply transitively on the set of triples of distinct points of $\mathbb{P}^1$. So to fix a decomposition in the family, we select a triple of points in each space. We focus on $\mathbb{P}U$. Call the points $[u_1],[u_2],[u_3]$. Then these determine three points in $\mathbb{P}U^*$, $[u^{1\perp}],[u^{2\perp}],[u^{3\perp}]$. We choose representatives u_1,u_2,u_3 satisfying $u_1+u_2+u_3=0$. We could have taken any linear relation, it just would introduce coefficients in the decomposition. We take the most symmetric relation to keep all three points on an equal footing. Similarly, we fix the scales on the $u^{j\perp}$ by requiring $u^{j\perp}(u_{j-1})=1$ and $u^{j\perp}(u_{j+1})=-1$, where indices are considered mod $\mathbb{Z}_3$, so $u_{3+1}=u_1$ and $u_{1-1}=u_3$.

In comparison with what we had before, letting the old indices be hatted, we have $\hat{u}_1=u_1$, $\hat{u}_2=u_2$, $\hat{u}_3=-u_3$ and $\hat{u}^1=u^{2\perp}$, $\hat{u}^2=-u^{1\perp}$, and $\hat{u}^3=-u^{3\perp}$. The effect is to make the symmetries of the decomposition more transparent. Our identifications of the ordered triples $\{u_1,u_2,u_3\}$ and $\{v_1,v_2,v_3\}$ exactly determine a linear isomorphism $a_0: U\to V$, and similarly for the other pairs of vector spaces. Note that $a_0=v_j\otimes u^{j+1\perp}+v_{j+1}\otimes u^{j+2\perp}$ for any $j=1,2,3$.

Then

$$
\begin{aligned}
(13)\qquad M_{\langle 2\rangle} = {} & a_0\otimes b_0\otimes c_0 \\
& + \langle (v_2\otimes u^{1\perp})\otimes(w_1\otimes v^{3\perp})\otimes(u_3\otimes w^{2\perp})\rangle_{\mathbb{Z}_3} \\
& + \langle (v_3\otimes u^{1\perp})\otimes(w_1\otimes v^{2\perp})\otimes(u_2\otimes w^{3\perp})\rangle_{\mathbb{Z}_3}.
\end{aligned}
$$

With this presentation, the $\mathfrak{S}_3\subset PGL_2\subset PGL_2^{\times 3}$ acting by permuting the indices transparently preserves the decomposition, with two orbits, the fixed point $a_0\otimes b_0\otimes c_0$ and the orbit of $(v_2\otimes u^{1\perp})\otimes(w_1\otimes v^{3\perp})\otimes(u_3\otimes w^{2\perp})$.

Remark 6.1. Note that here there are no nilpotent matrices appearing.

Remark 6.2. The geometric picture of the decomposition of $M_{\langle 2\rangle}$ can be rephrased as follows. Consider the space of linear isomorphisms $U\to V$ (mod scalar multiplication) as the projective space $\mathbb{P}^3$ of 2×2 matrices, in which we fix coordinates, coming from the choice of basis for U,V. The choice of basis also determines an identification between U and V. Then a_0 represents in $\mathbb{P}^3$ a point of rank 2, which can be taken as the identity in the choice of coordinates. The other 6 points $Q_i=u_i\otimes u^{j\perp}$ appearing in the first factor of the decomposition can be determined as follows. The points $P_i=u_i\otimes u^{i\perp}$ (in the identification) represent the choice of 3 points in the conic obtained by cutting with a plane (e.g. the plane of traceless matrices) the quadric $q=Seg(\mathbb{P}^1\times\mathbb{P}^1)$ of matrices of rank 1. Through each P_i one finds lines of the two rulings of q, call then L_i,M_i. Then the six points Q_i are given by:

$$Q_1=L_1\cap M_2,\ Q_2=L_2\cap M_3,\ Q_3=L_3\cap M_1$$

$$Q_4=M_1\cap L_2,\ Q_5=M_2\cap L_3,\ Q_6=M_3\cap L_1.$$

An analogue of the construction determines the seven points in the other two factors of the tensor product, so that the 7 final summands can be determined combinatorially and the $\mathbb{Z}_2,\mathbb{Z}_3$ symmetries can be easily recognized.

The geometric construction can be generalized to higher dimensional spaces, so it could insight for extensions to larger matrix multiplication tensors. The difficult part is to determine how one should combine the points constructed in each factor of the tensor product, in order to produce a decomposition of $M_{\langle n\rangle}$.

When we view (8) projectively, we get

$$M_{\langle 2\rangle} = (v_1u^{1\perp} + v_2u^{3\perp})\otimes(w_1v^{1\perp} + w_2v^{3\perp})\otimes(u_1w^{1\perp} + u_2w^{3\perp}) \tag{14}$$
$$+ v_1u^{2\perp}\otimes w_1v^{2\perp}\otimes u_1w^{2\perp} \tag{15}$$
$$+ v_3u^{1\perp}\otimes w_3v^{1\perp}\otimes u_3w^{1\perp} \tag{16}$$
$$+ v_2u^{3\perp}\otimes w_2v^{3\perp}\otimes u_2w^{3\perp} \tag{17}$$
$$\langle v_1u^{1\perp}\otimes w_2v^{2\perp}\otimes u_3w^{3\perp}\rangle_{\mathbb{Z}_3}. \tag{18}$$

With this presentation, $S_3 \subset \Gamma'_S$ is again transparent.

6.2. Symmetries of Γ_{Str}. Let $M_{\langle n\rangle} = \sum_{j=1}^{r} t_j$ be a rank decomposition for $M_{\langle n\rangle}$ and write $t_j = a_j\otimes b_j\otimes c_j$. Let $r_j = (r_{A,j}, r_{B,j}, r_{C,j}) := (\mathrm{rank}(a_j), \mathrm{rank}(b_j), \mathrm{rank}(c_j))$, and let $\tilde{r}_j$ denote the unordered triple. The following proposition is clear:

Proposition 6.3. Let S be a rank decomposition of $M_{\langle n\rangle}$. Partition S by un-ordered rank triples into disjoint subsets: $S := \{S_{1,1,1}, S_{1,1,2}, \dots, S_{n,n,n}\}$. Then Γ'_S preserves each $S_{s,t,u}$.

We can say more about rank one elements:

If $a \in U^*\otimes V$ and $\mathrm{rank}(a) = 1$, then there are unique points $[\mu] \in \mathbb{P}U^*$ and $[v] \in \mathbb{P}V$ such that $[a] = [\mu\otimes v]$.

Now given a decomposition S of $M_{\langle n\rangle}$, define $S_{U^*} \subset \mathbb{P}U^*$ and $S_U \subset \mathbb{P}U$ to correspond to the elements appearing in $S_{1,1,1}$. Then Γ'_S preserves S_U and S_{U^*}.

In the case of Strassen's decomposition Str_U is a configuration of three points in $\mathbb{P}^1$, so a priori we must have $\Gamma'_{Str} \cap PGL(U) \subset S_3$. If we insist on the standard $\mathbb{Z}_3$-symmetry (i.e., restrict to the subfamily of decompositions where there is a standard cyclic symmetry), there is just one PGL_2 and we have $\Gamma'_{Str} \subseteq S_3$. Recall that this is no loss of generality as the full symmetry group is the same for all decompositions in the family. We conclude $\Gamma_{Str} \subseteq S_3 \times D_3$. We have already seen $S_3 \times \mathbb{Z}_3 \subset \Gamma_{Str}$, Burichenko [3] shows that in addition there is a non-convenient $\mathbb{Z}_2$ obtained by taking the convenient $\mathbb{Z}_2$ (which sends the decomposition to another decomposition in the family) and then conjugating by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \subset PGL_2 \subset PGL_2^{\times 3}$ which sends the decomposition back to Str. We recover (with a new proof of the upper bound) Burichenko's theorem:

Theorem 6.4. [3] The symmetry group of Strassen's decomposition of $M_{\langle 2\rangle}$ is $S_3 \times D_3 \subset PGL_2^{\times 3} \times D_3 = G_{M_{\langle 2\rangle}}$.

7. How to prove Strassen's decomposition is actually matrix multiplication

The group Γ_{Str} acts on $(U^*\otimes U)^{\otimes 3}$ (in different ways, depending on the choice of decomposition in the family). Say we did not know Str but did know its symmetry group. Then we could look for it inside the space of Γ_{Str} invariant tensors. In future work we plan to take candidate symmetry groups for matrix multiplication decompositions and look for decompositions with elements from these subspaces. In this paper we simply illustrate the idea by going in the other direction: furnishing a proof that Str is a decomposition of $M_{\langle 2\rangle}$, by using the invariants to reduce the computation to a simple verification. We accomplish this in §7.2 below. We first give yet another proof that Strassen's decomposition is matrix multiplication using the fact that $M_{\langle 2\rangle}$ is characterized by its symmetries.

7.1. Proof that Strassen's algorithm works via characterization by symmetries. Here is a proof that illustrates another potentially useful property of $M_{\langle n\rangle}$: it is characterized by its symmetry group [6] Any $T \in (U^*\otimes V)\otimes(V^*\otimes W)\otimes(W^*\otimes U)$ that is invariant under $PGL(U) \times PGL(V) \times PGL(W) \ltimes D_3$ is up to scale to $M_{\langle n\rangle}$. Any $T \in (U^*\otimes V)\otimes(V^*\otimes W)\otimes(W^*\otimes U)$ that is invariant under a group isomorphic to $PGL(U) \times PGL(V) \times PGL(W) \ltimes D_3$ is $GL(A) \times GL(B) \times GL(C) \times S_3$-equivalent up to scale to $M_{\langle n\rangle}$.

Remark 7.1. $M_{\langle n\rangle}$ is also characterized as a polynomial by its symmetry group $\tilde{G}_{M_{\langle n\rangle}}$, and any $T \in (U^*\otimes V)\otimes(V^*\otimes W)\otimes(W^*\otimes U)$ that is invariant under $PGL(U) \times PGL(V) \times PGL(W)$ is up to scale to $M_{\langle n\rangle}$. However, it is not characterized up to $GL(A) \times GL(B) \times GL(C)$-equivalence by $G'_{M_{\langle n\rangle}}$ in the strong sense of up to isomorphism because $(X, Y, Z) \mapsto \mathrm{trace}(YXZ)$ has an isomorphic symmetry group but is not $GL(A) \times GL(B) \times GL(C)$-equivalent.

By the above discussion, we only need to check the right hand side of (13) is invariant under $PGL(U) \times PGL(V) \times PGL(W)$ and to check its scale. But by symmetry, it is sufficient to check it is invariant under $PGL(U)$. For this it is sufficient to check it is annihilated by $\mathfrak{sl}(U)$, and again by symmetry, it is sufficient to check it is annihilated by $u_1\otimes u^{1\perp}$, which is a simple calculation.

7.2. Spaces of invariant tensors. As an S_3-module $A = U^*\otimes V = [21]\otimes[21] = [3]\oplus[21]\oplus[1^3]$. In what follows we use the decompositions:

$$S^2[21] = [3] \oplus [21]$$
$$\Lambda^2[21] = [1^3]$$
$$S^3[21] = [3] \oplus [21] \oplus [1^3].$$

The space of standard cyclic Z_3-invariant tensors in $A^{\otimes 3} = S^3A\oplus S_{21}A^{\oplus 2}\oplus\Lambda^3A$ is $S^3A\oplus\Lambda^3A$. Inside the space of Z_3-invariant vectors we want to find instances of the trivial S_3-module [3] in $S^3([3] \oplus [2,1] \oplus [1^3]) \oplus \Lambda^3([3] \oplus [2,1] \oplus [1^3])$. We have

$$\begin{aligned} S^3([3] \oplus [2,1] \oplus [1^3]) = {} & S^3[3] \oplus S^2[3]\otimes[2,1] \oplus S^2[3]\otimes[1^3] \oplus [3]\otimes S^2[2,1] \oplus [3]\otimes[21]\otimes[1^3] \\ & \oplus [3]\otimes S^2[1^3]\otimes S^3[21] \oplus S^2[21]\otimes[13] \oplus [21]\otimes S^2[1^3] \oplus S^3[1^3] \end{aligned}$$

and four factors contain (or are) a trivial representation: $S^3[3], [3]\otimes S^2[2,1], [3]\otimes S^2[1^3], S^3[21]$ Similarly

$$\Lambda^3([3] \oplus [21] \oplus [1^3]) = \Lambda^2[21]\otimes[3] \oplus \Lambda^2[21]\otimes[1^3] \oplus [3]\otimes[21]\otimes[1^3]$$

of which $\Lambda^2[21]\otimes[1^3]$ is the unique trivial submodule.

In summary:

Proposition 7.2. The space of $S_3 \times Z_3$ invariants in $(U^*\otimes U)^{\otimes 3}$ when $\dim U = 2$ is five dimensional.

By a further direct calculation we obtain:

Proposition 7.3. The space of $S_3 \times D_3$ invariants in $(U^*\otimes U)^{\otimes 3}$ when $\dim U = 2$ is four dimensional.

So if we knew there were an $S_3 \times D_3$ invariant decomposition of $M_{\langle 2\rangle}$, it would be a simple calculation to find it as a linear combination of four basis vectors of the $S_3 \times D_3$-invariant tensors. In future work we plan to assume similar invariance for larger matrix multiplication tensors to shrink the search space to manageable size.

References

1. Grey Ballard, Christian Ikenmeyer, J.M. Landsberg, and Nick Ryder, The geometry of rank decompositions of matrix multiplication ii: 3x3 matrices, preprint.
2. H. Bermudez, S. Garibaldi, and V. Larsen, Linear preservers and representations with a 1-dimensional ring of invariants, Trans. Amer. Math. Soc. 366 (2014), no. 9, 4755–4780. MR 3217699
3. Vladimir P. Burichenko, On symmetries of the strassen algorithm, CoRR abs/1408.6273 (2014).
4. ______, Symmetries of matrix multiplication algorithms. I, CoRR abs/1508.01110 (2015).
5. Hans F. de Groote, On varieties of optimal algorithms for the computation of bilinear mappings. I. The isotropy group of a bilinear mapping, Theoret. Comput. Sci. 7 (1978), no. 1, 1–24. MR 0506377
6. Fulvio Gesmundo, Geometric aspects of iterated matrix multiplication, J. Algebra 461 (2016), 42–64. MR 3513064
7. J. M. Landsberg, Tensors: geometry and applications, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. MR 2865915
8. Volker Strassen, Gaussian elimination is not optimal, Numer. Math. 13 (1969), 354–356. MR 40 # 2223

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