Proposition 11. $\Gamma(A;B)$ is a boolean lattice, a sublattice of the lattice $\mathscr{P}(A\times B)$.

Proof. That it's a sublattice is obvious. That it has complement, is also obvious. Distributivity follows from distributivity of $\mathscr{P}(A\times B)$. □

4 Main part

Theorem 12. Let A, B be sets. The following are mutually reverse order isomorphisms between $\mathfrak{F}(\Gamma(A;B))$ and $\mathsf{FCD}(A;B)$:

1. $\mathcal{A}\mapsto\bigsqcap^{\mathsf{FCD}}\mathcal{A}$;
2. $f\mapsto\operatorname{up}^{(\mathsf{FCD}(A;B);\Gamma(A;B))}f$.

Proof. Let's prove that $\operatorname{up}^{(\mathsf{FCD}(A;B);\Gamma(A;B))}f$ is a filter for every funcoid f. We need to prove that $P\cap Q\in\operatorname{up}f$ whenever

$$P=\bigcap_{i=0,\ldots,n-1}(X_i\times Y_i\cup\overline{X_i}\times B)\quad\text{and}\quad Q=\bigcap_{i=0,\ldots,m-1}(X_i'\times Y_i'\cup\overline{X_i'}\times B).$$

This follows from $P\in\operatorname{up}f\Leftrightarrow\forall i\in 0,\ldots,n-1\colon\langle f\rangle X_i\sqsubseteq Y_i$ and likewise for Q, so having $\langle f\rangle(X_i\cap X_j')\sqsubseteq Y_i\cap Y_j'$ for every pair $(i;j)$. From this it follows that $P\cap Q\in\operatorname{up}f$. [TODO: more detailed proof of this]

[TODO: More detailed proof in both directions.]

Let $\mathcal{A}$, $\mathcal{B}$ be filters on Γ. Let $\bigsqcap^{\mathsf{FCD}}\mathcal{A}=\bigsqcap^{\mathsf{FCD}}\mathcal{B}$. We need to prove $\mathcal{A}=\mathcal{B}$. (The rest follows from the theorem 6.104 from my book [1]). We have:

$$\begin{aligned}
\mathcal{A}=\bigsqcap\{X\times Y\cup\overline{X}\times B\in\mathcal{A}\mid X\in\mathscr{P}A,Y\in\mathscr{P}B\} &=\\
\bigsqcap\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\exists A\in\mathcal{A}\colon P\subseteq X\times Y\cup\overline{X}\times B\} &=\\
\bigsqcap\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\exists P\in\mathcal{A}\colon\langle A\rangle^*X\subseteq Y\} &= (*)\\
\bigsqcap\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\bigsqcap\{\langle P\rangle^*X\mid A\in\mathcal{A}\}\sqsubseteq Y\} &=\\
\bigsqcap\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\bigsqcap\{\langle P\rangle^*X\mid A\in\upuparrows\mathcal{A}\}\sqsubseteq Y\} &=\\
\bigsqcap\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\langle(\mathsf{FCD})\upuparrows\mathcal{A}\rangle X\sqsubseteq Y\} &= (**)\\
\bigsqcap\left\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\left\langle\bigsqcap^{\mathsf{FCD}}\upuparrows\mathcal{A}\right\rangle X\sqsubseteq Y\right\} &=\\
\bigsqcap\left\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\left\langle\bigsqcap^{\mathsf{FCD}}\mathcal{A}\right\rangle X\sqsubseteq Y\right\}&.
\end{aligned}$$

(*) by properties of generalized filter bases, because $\{\langle P\rangle^*X\mid P\in\mathcal{A}\}$ is a filter base.

(**) by theorem 8.3 in [1].

Similarly

$$\mathcal{B}=\bigsqcap\left\{X\times Y\cup\overline{X}\times B\mid X\in\mathscr{P}A,Y\in\mathscr{P}B,\left\langle\bigsqcap^{\mathsf{FCD}}\mathcal{B}\right\rangle X\sqsubseteq Y\right\}.$$

Thus $\mathcal{A}=\mathcal{B}$. □

[TODO: The above bijection preserves composition?]

[TODO: Which properties of funcoids follow?] [TODO: Specifically, what about relationships with reloids?] [TODO: What about analogs of reloids properties?]

[TODO: properties of the filtrator $(\mathsf{FCD}(A;B);\Gamma(A;B))$]

[TODO: For pointfree funcoids?]

Proposition 13. $\upuparrows$ and $\downdownarrows$ are mutually inverse bijections between $\mathfrak{F}(\Gamma(A;\,B))$ and funcoidal reloids.