Analysis of Large Graphs: Community Detection

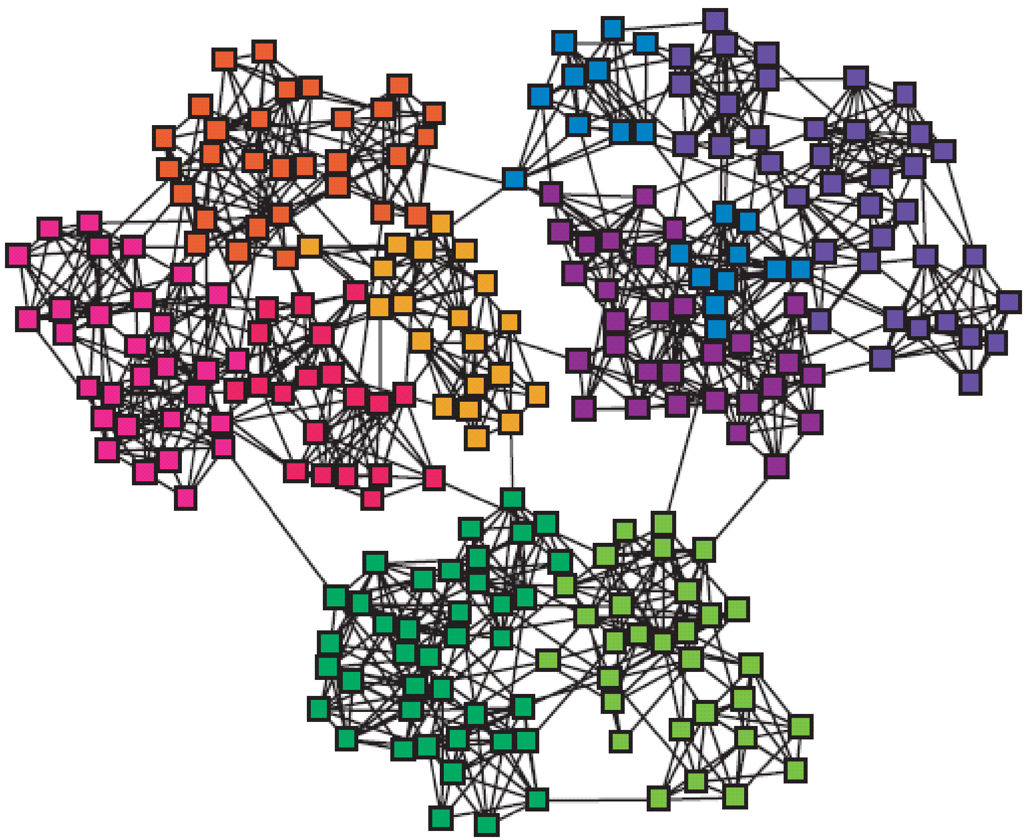
Mining of Massive Datasets

Jure Leskovec, Anand Rajaraman, Jeff Ullman
Stanford University

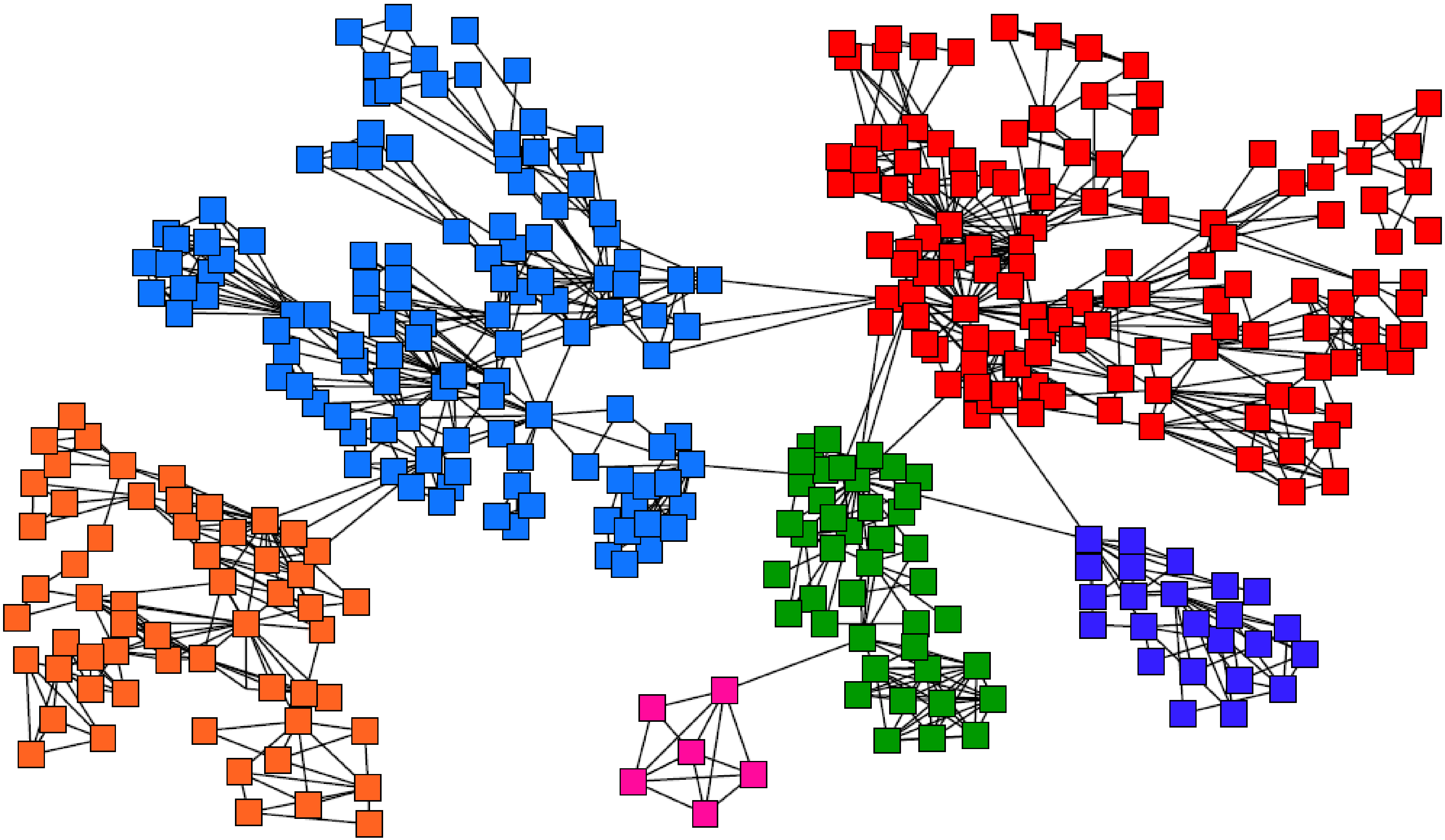
http://www.mmds.org

Networks & Communities

- **We often think of networks being organized into modules, cluster, communities:**

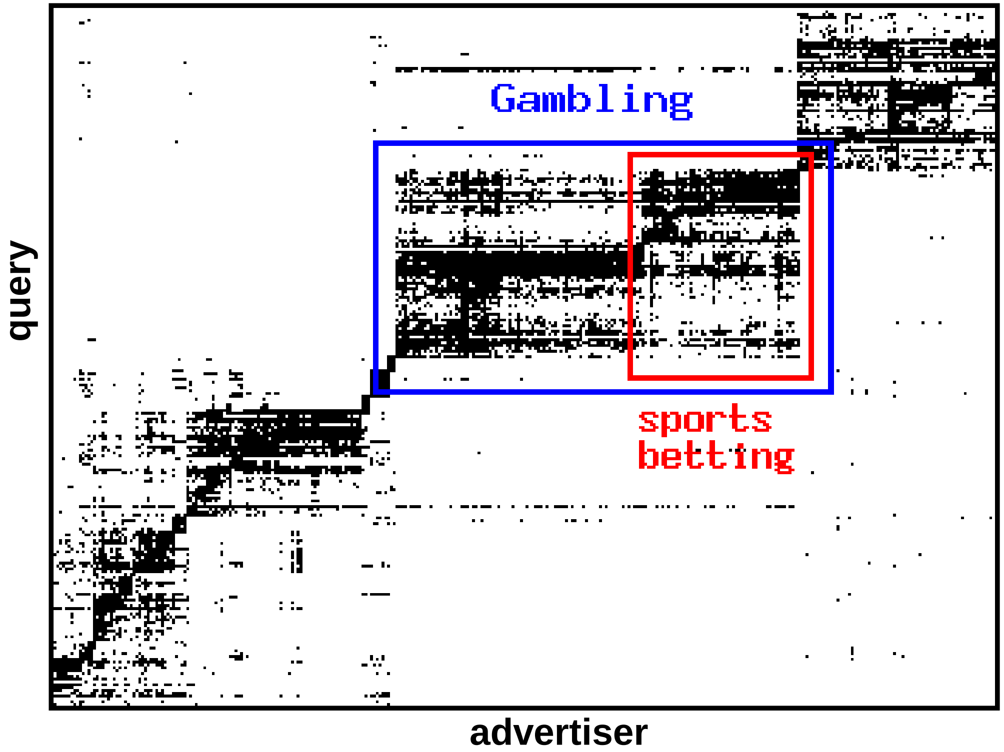


Goal: Find Densely Linked Clusters



Micro-Markets in Sponsored Search

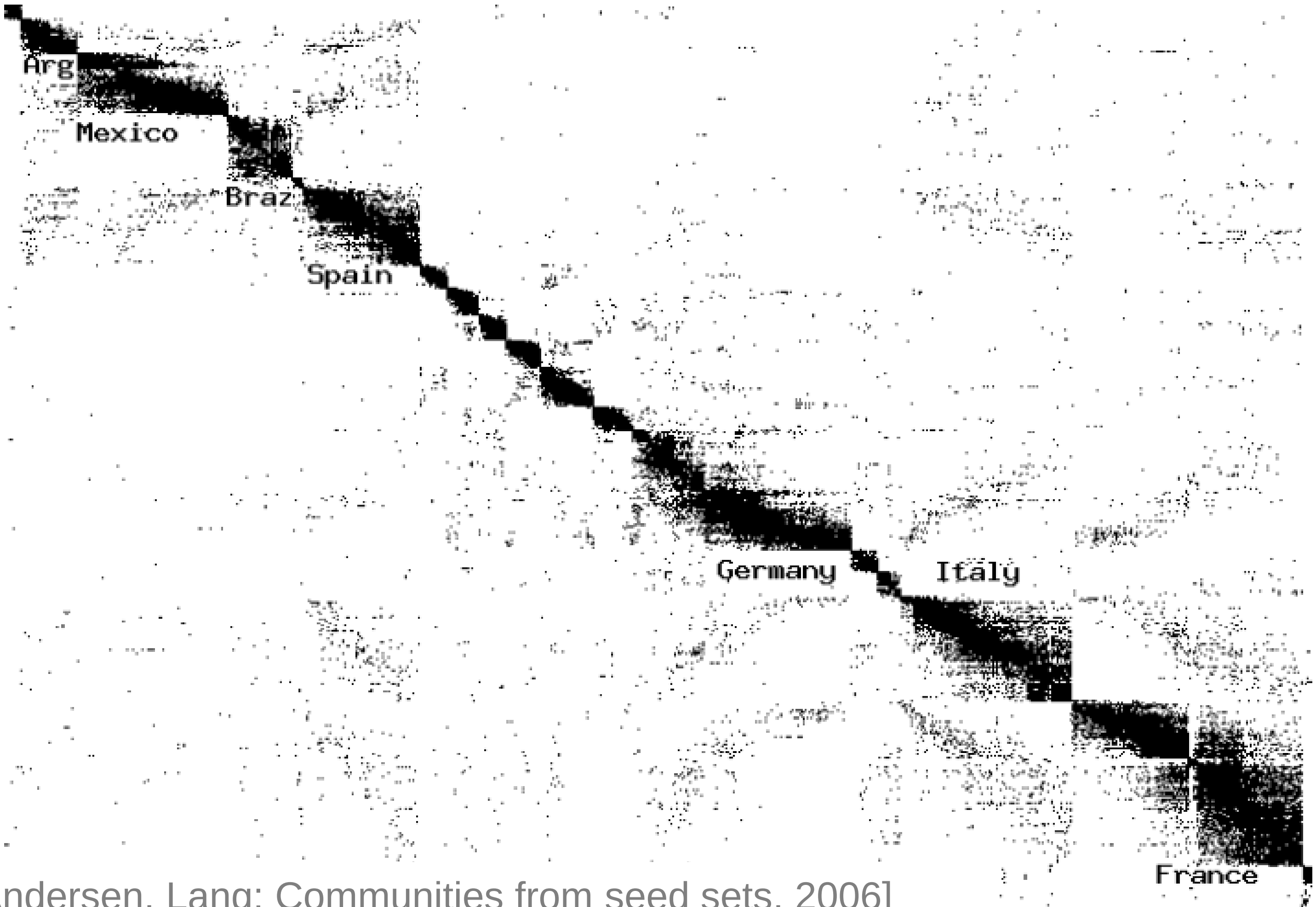
- **Find micro-markets by partitioning the query-to-advertiser graph:**



[Andersen, Lang: Communities from seed sets, 2006]

Movies and Actors

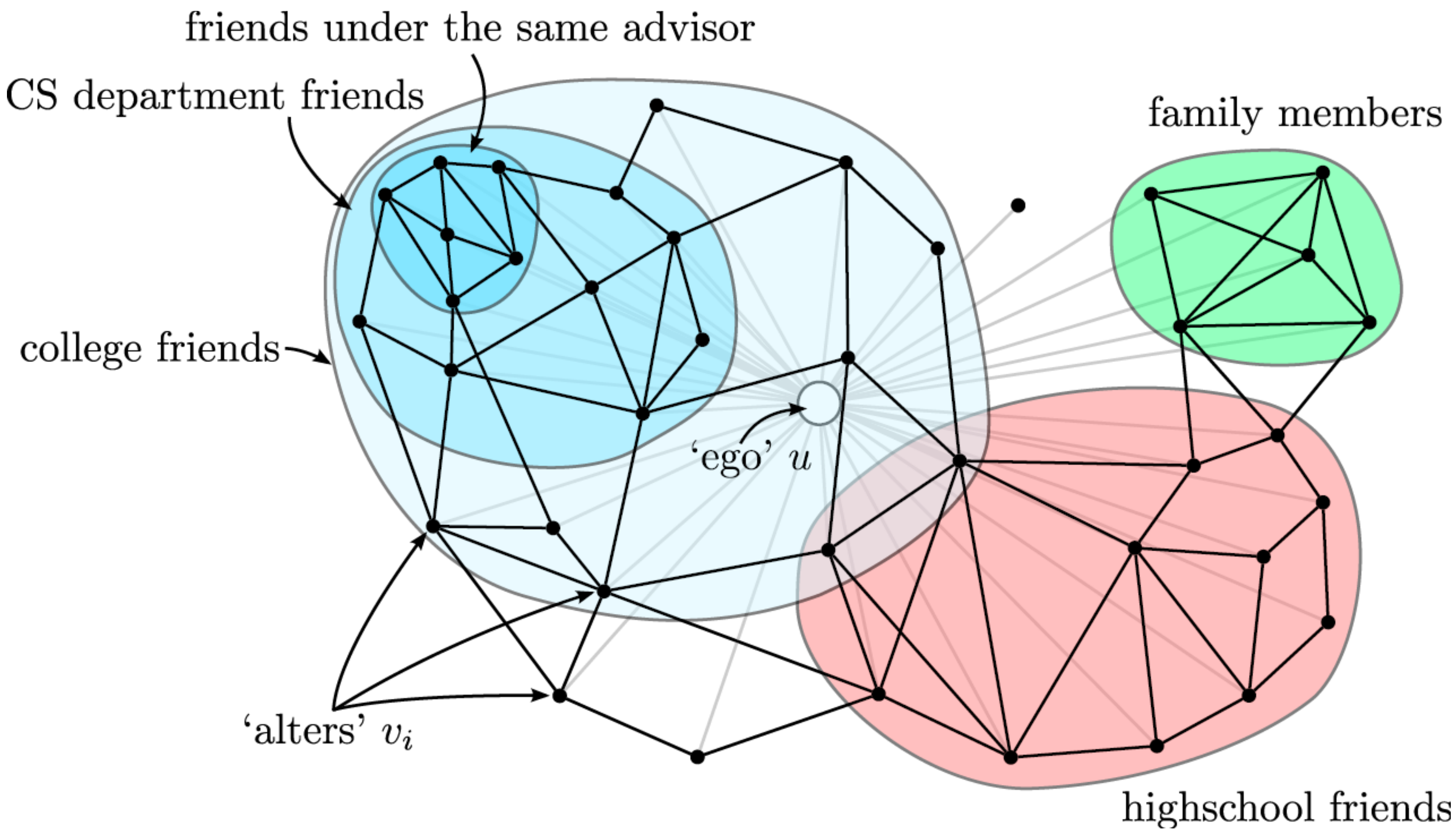
- **Clusters in Movies-to-Actors graph:**



[Andersen, Lang: Communities from seed sets, 2006]

Twitter & Facebook

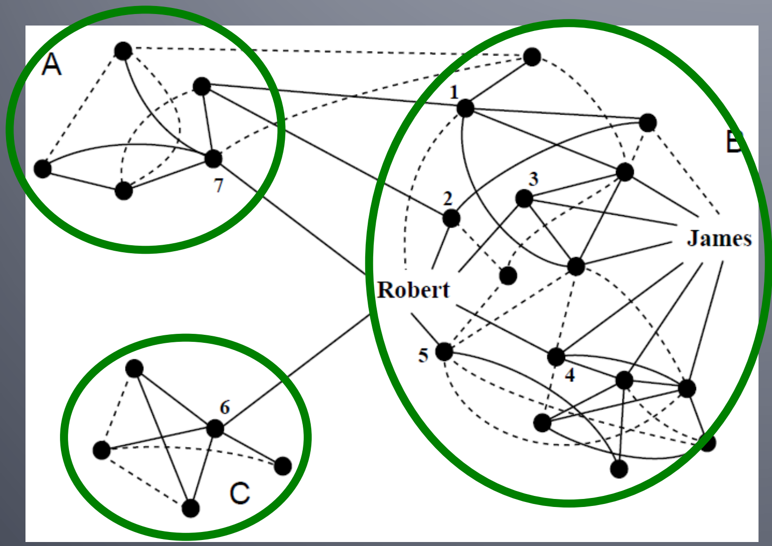
- **Discovering social circles, circles of trust:**

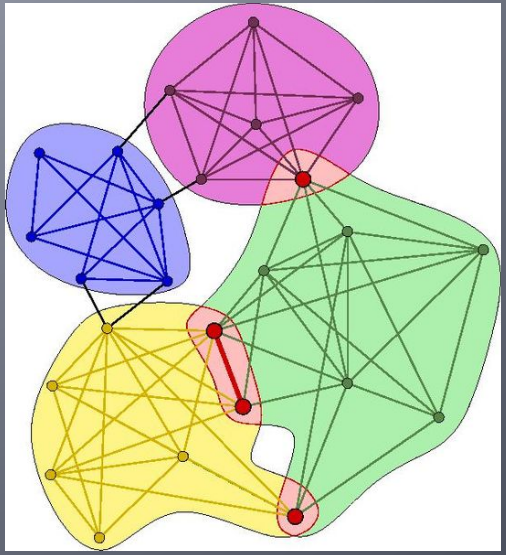


[McAuley, Leskovec: Discovering social circles in ego networks, 2012]

Community Detection

How to find communities?

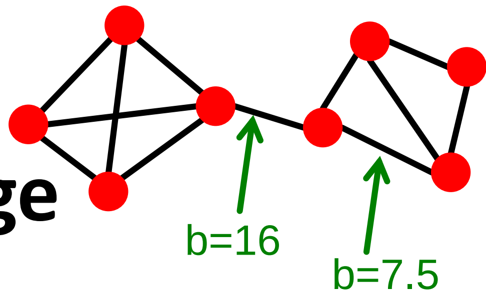


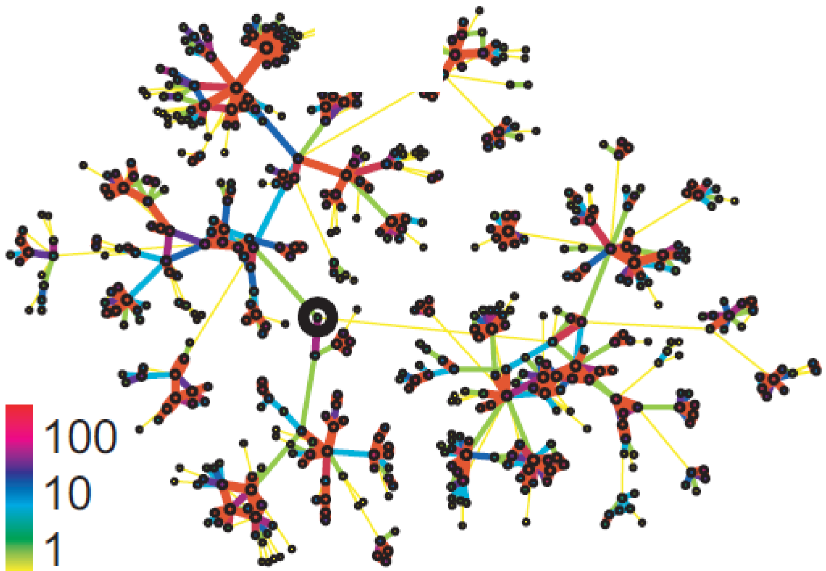


We will work with **undirected** (unweighted) networks

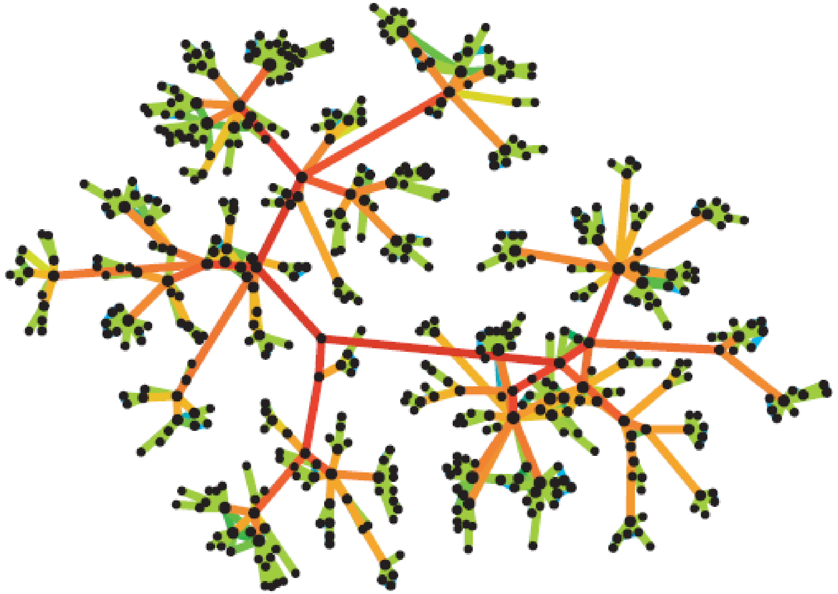
Method 1: Strength of Weak Ties

- **Edge betweenness:** Number of shortest paths passing over the edge
- **Intuition:**





Edge strengths (call volume) in a real network



Edge betweenness in a real network

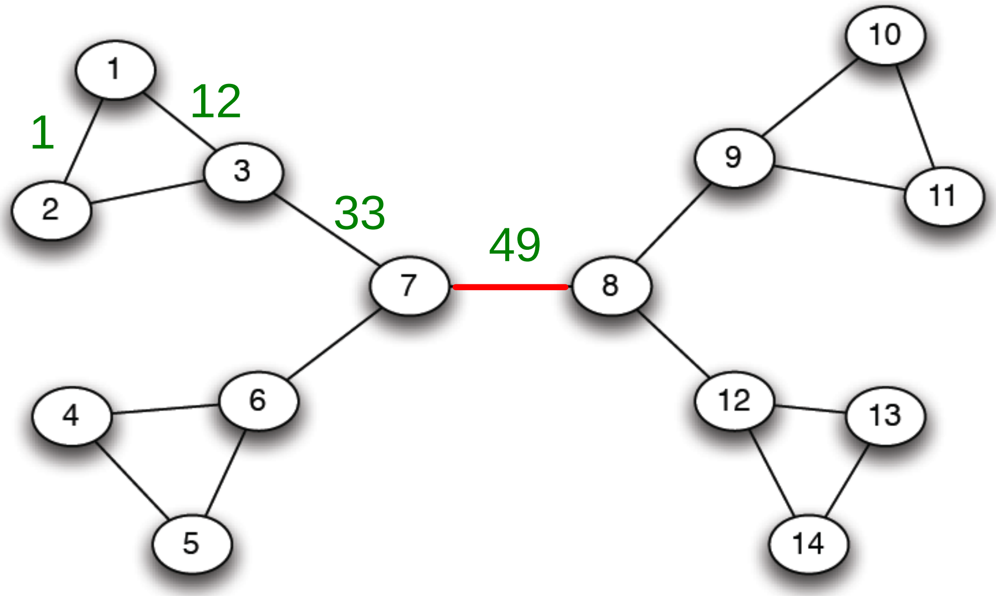
Method 1: Girvan-Newman

- Divisive hierarchical clustering based on the notion of edge **betweenness:**

  Number of shortest paths passing through the edge

- **Girvan-Newman Algorithm:**
    - **Undirected unweighted networks**
  - **Repeat until no edges are left:**
    - Calculate betweenness of edges
    - Remove edges with highest betweenness
  - Connected components are communities
  - Gives a hierarchical decomposition of the network

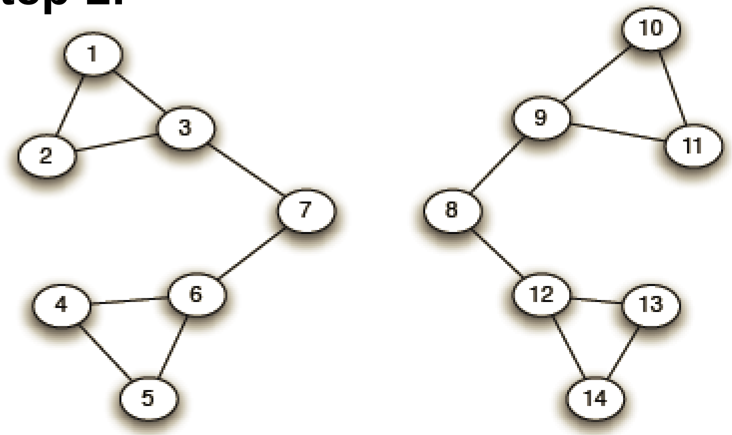
Girvan-Newman: Example



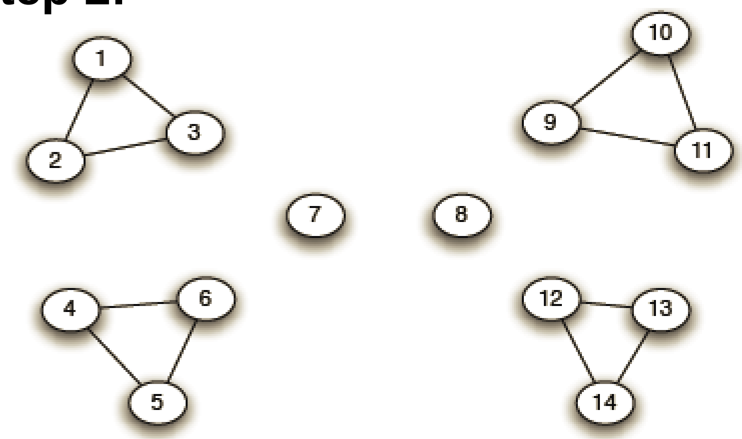
Need to re-compute betweenness at every step

Girvan-Newman: Example

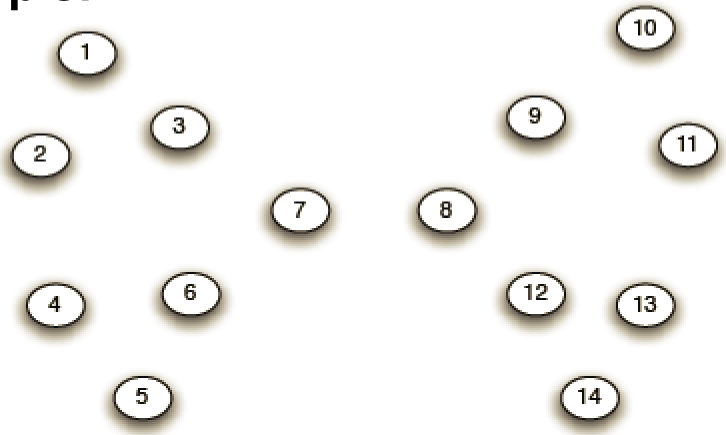
Step 1:



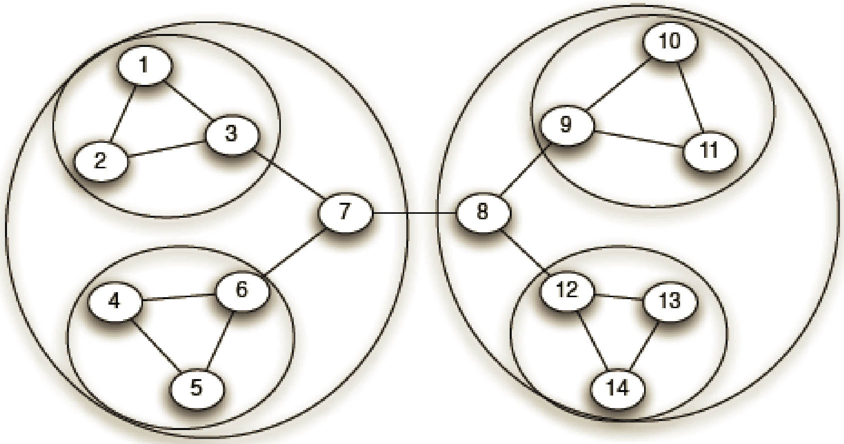
Step 2:



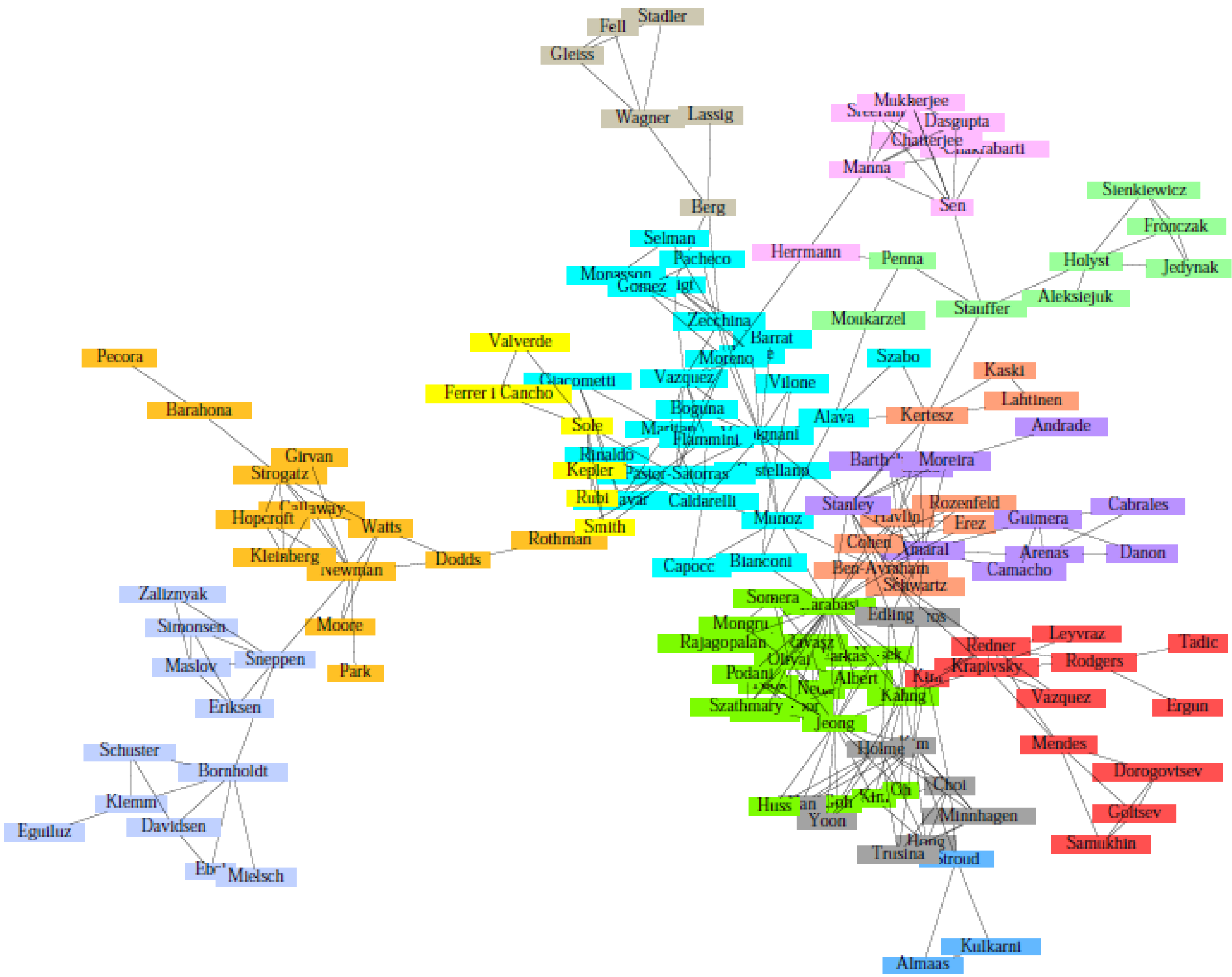
Step 3:



Hierarchical network decomposition:



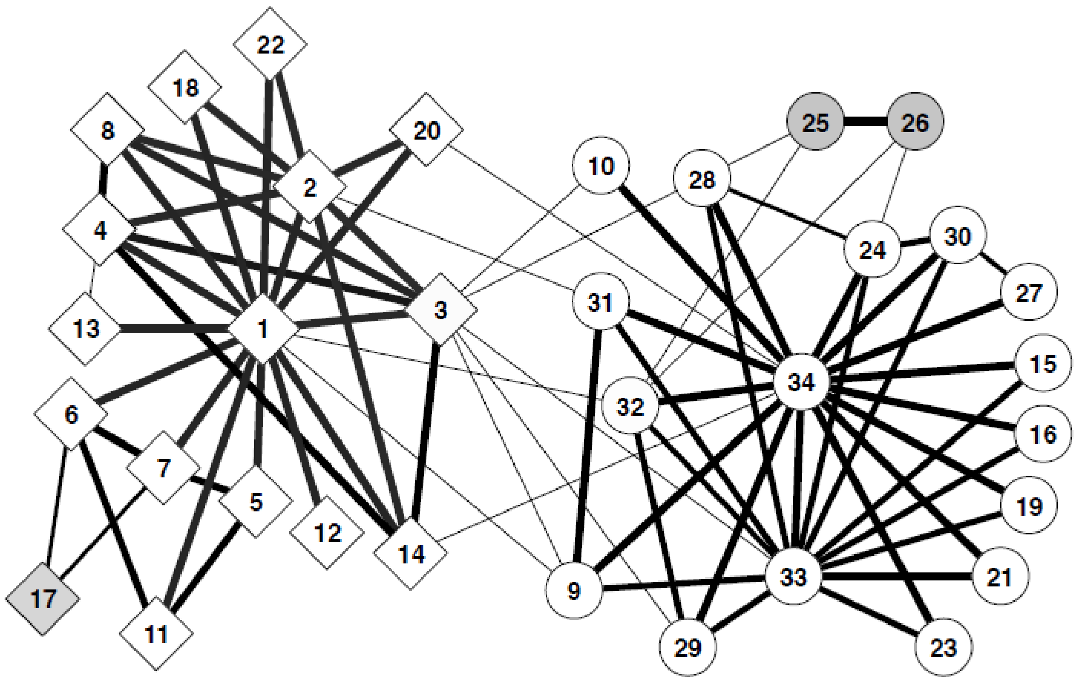
Girvan-Newman: Results

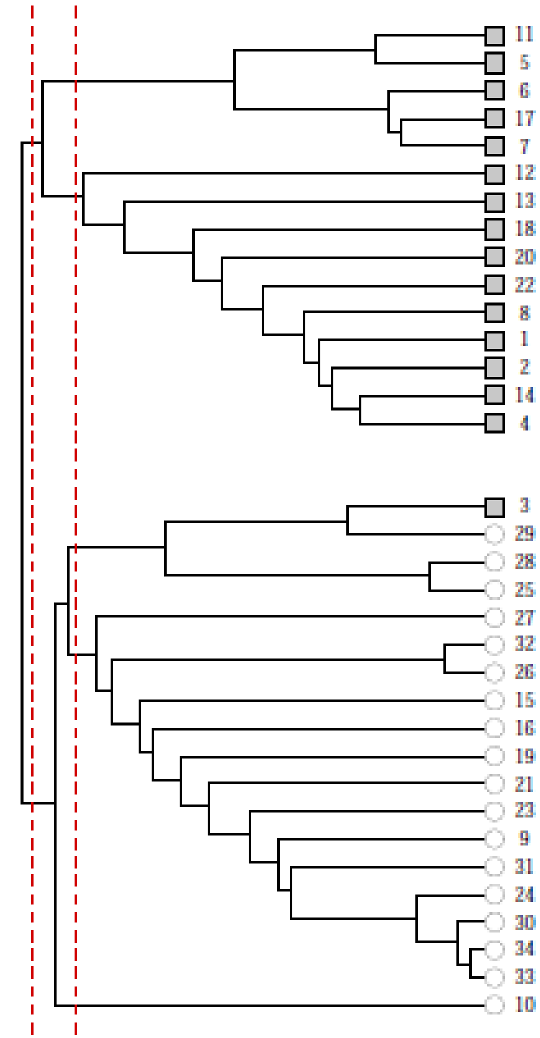


Communities in physics collaborations

Girvan-Newman: Results

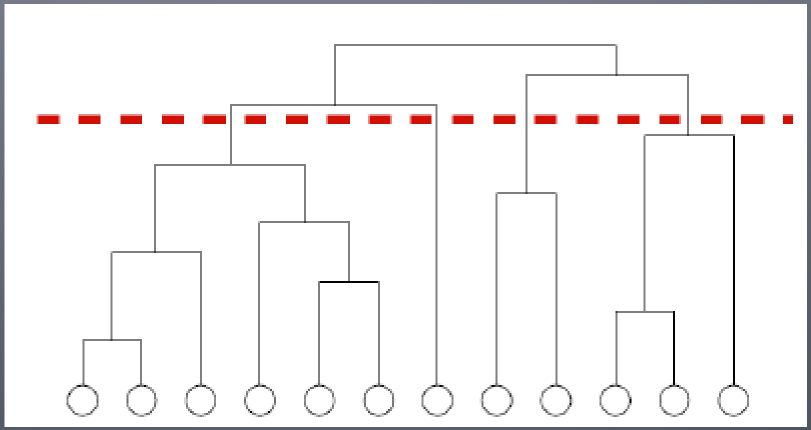
- **Zachary's Karate club:** Hierarchical decomposition





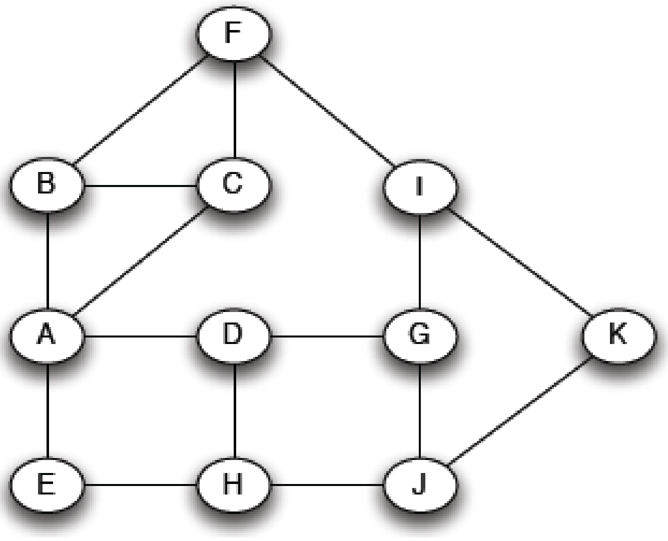
We need to resolve 2 questions

1. **How to compute betweenness?**
2. **How to select the number of clusters?**

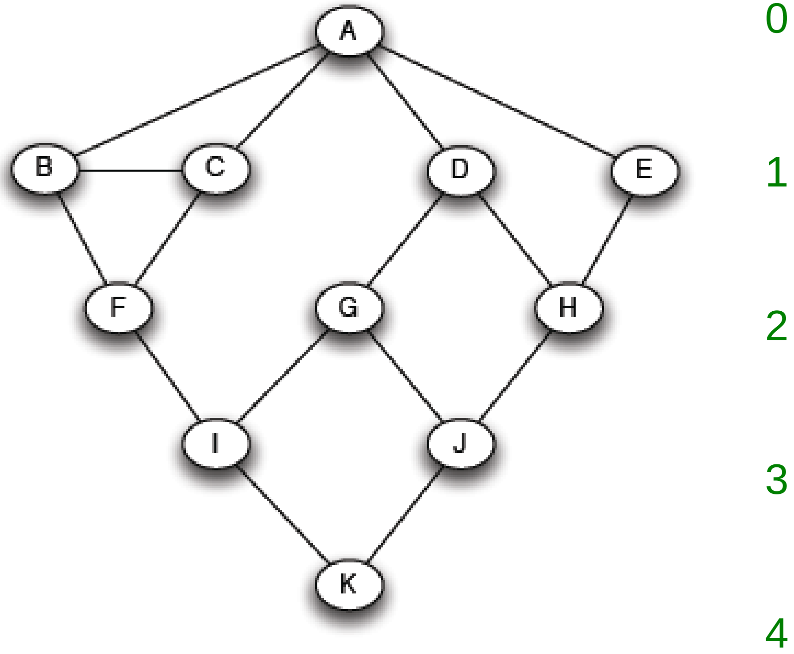


How to Compute Betweenness?

- **Want to compute betweenness of paths starting at node A**

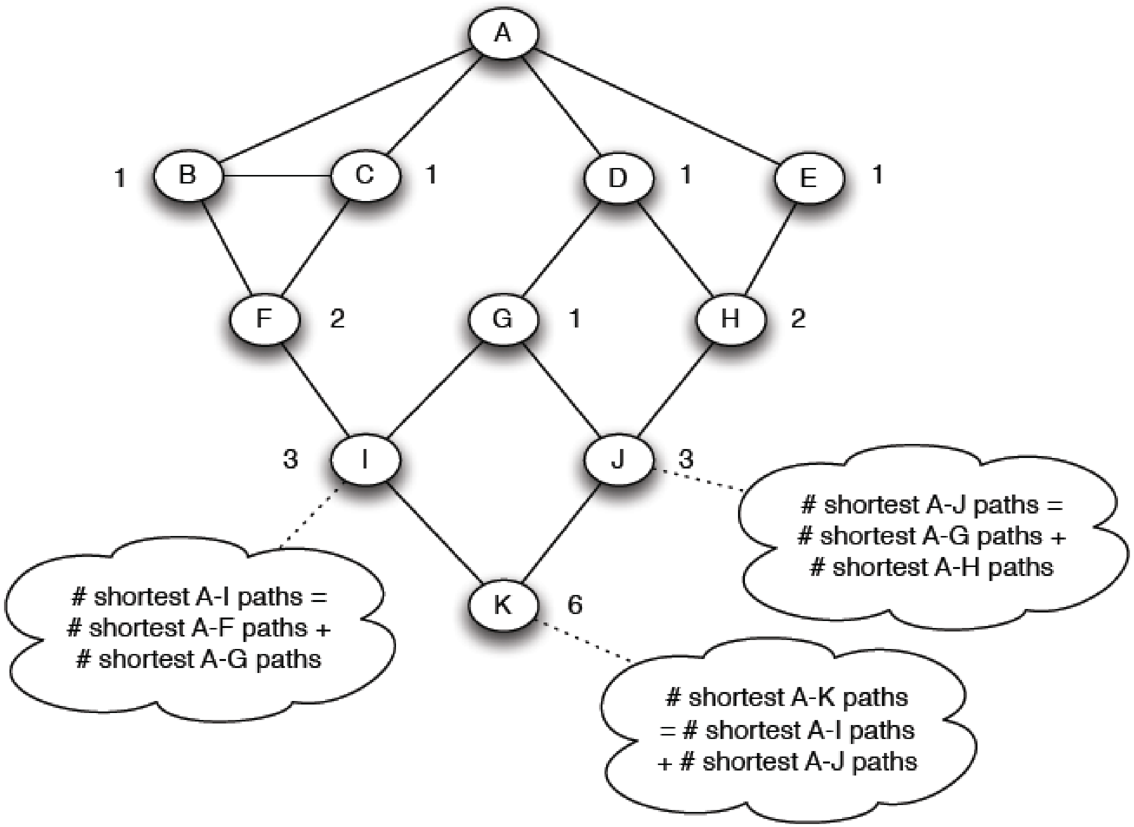


- **Breath first search starting from A:**



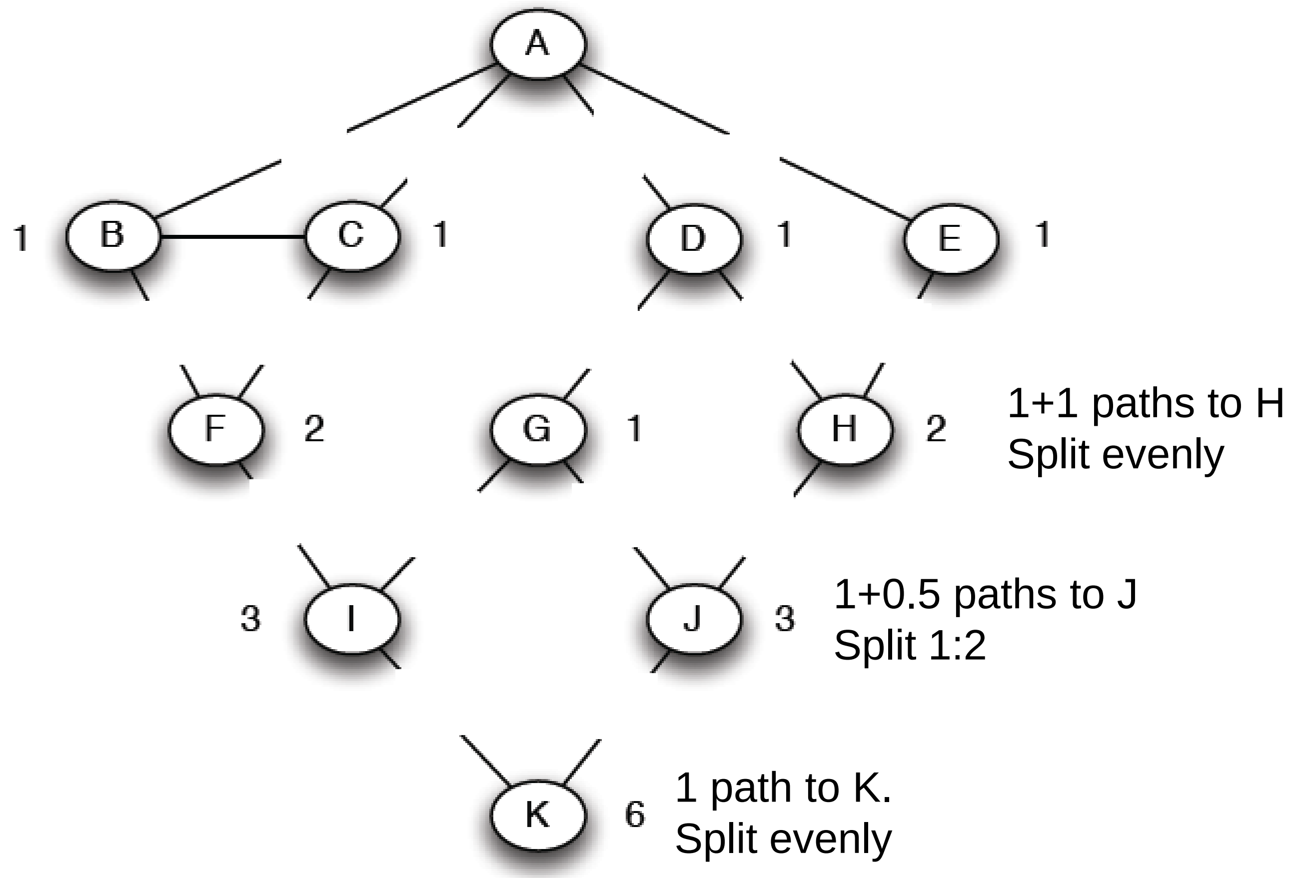
How to Compute Betweenness?

- **Count the number of shortest paths from A to all other nodes of the network:**



How to Compute Betweenness?

- **Compute betweenness by working up the tree:** If there are multiple paths count them fractionally



The algorithm:
•Add edge **flows**:
-- node flow = 1+∑child edges
-- split the flow up based on the parent value
• Repeat the BFS procedure for each starting node U

How to Compute Betweenness?

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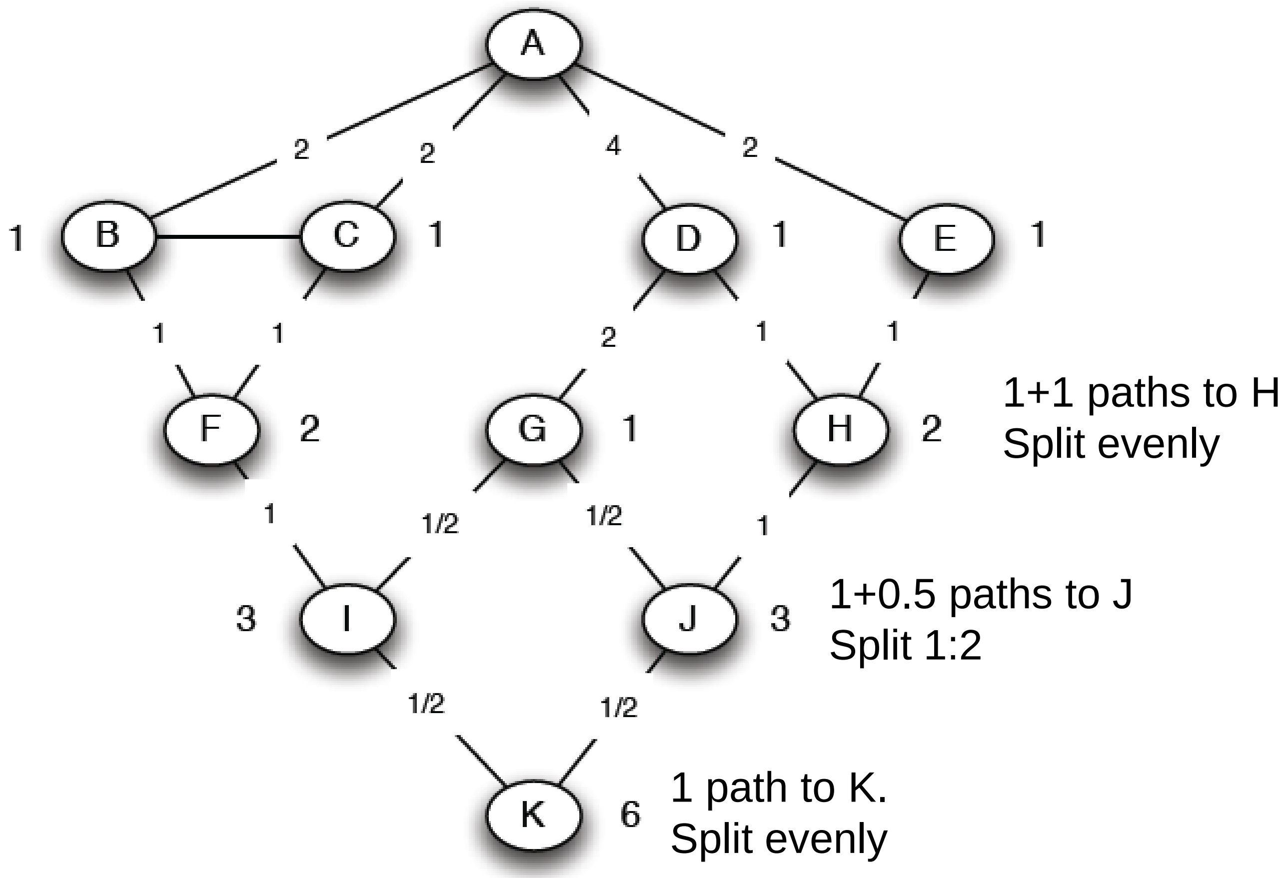

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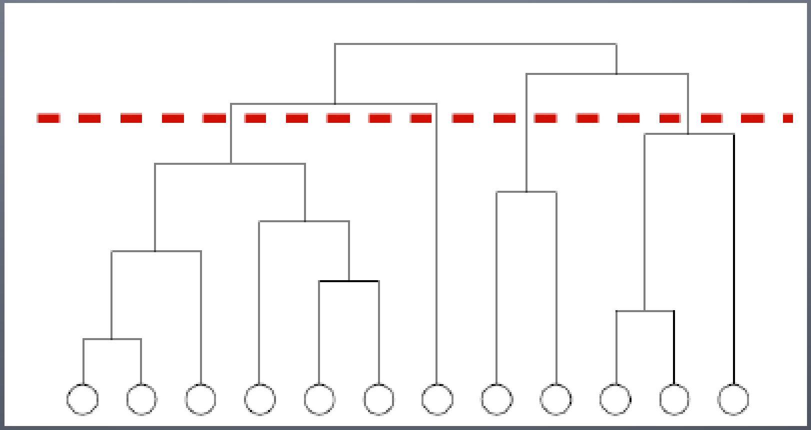
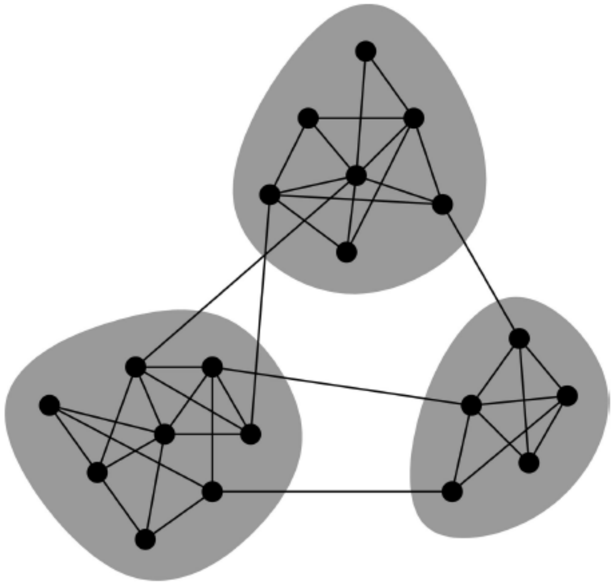


Network Communities

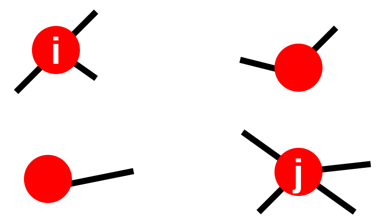
- **Communities:** sets of **tightly connected nodes**
- Define: **Modularity Q**
  - A measure of how well a network is partitioned into communities
  - Given a partitioning of the network into groups $s \in S$:



$$Q \;\propto\; \sum_{s \in S} [\, (\#\text{ edges within group } s) - \underbrace{(\text{expected }\#\text{ edges within group } s)\,]}_{\text{Need a null model!}}$$

Null Model: Configuration Model

- **Given real G on n nodes and m edges, construct rewired network G'**
  - Same degree distribution but random connections
  - Consider G' as a multigraph
  - **The expected number of edges between nodes i and j** of degrees k_i and k_j equals to: $k_i \cdot \frac{k_j}{2m} = \frac{k_i k_j}{2m}$
    - The expected number of edges in (multigraph) **G'**:
      - $= \frac{1}{2}\sum_{i \in N}\sum_{j \in N}\frac{k_i k_j}{2m} = \frac{1}{2} \cdot \frac{1}{2m}\sum_{i \in N} k_i \left(\sum_{j \in N} k_j\right) =$
      - $= \frac{1}{4m} 2m \cdot 2m = m$



Note: $\sum_{u \in N} k_u = 2m$

Modularity

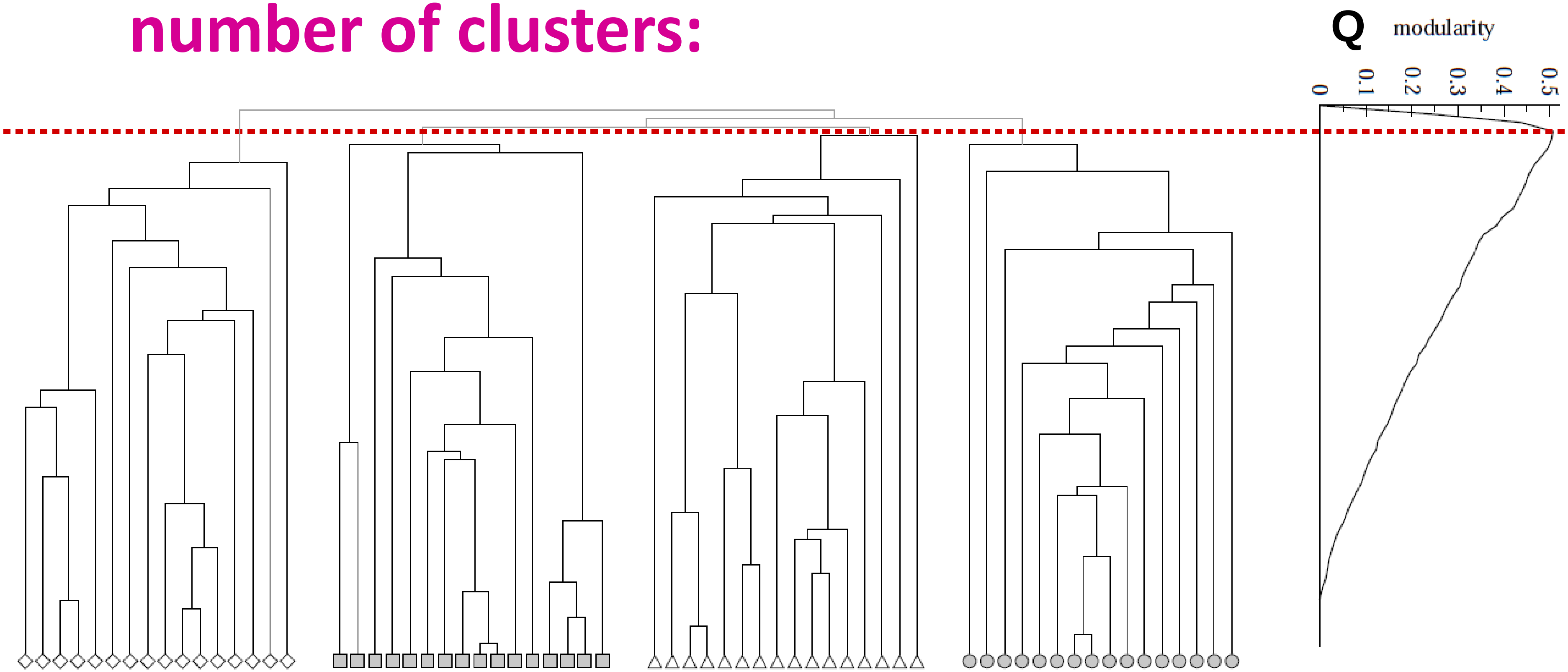
- **Modularity of partitioning S of graph G:**
  - **Q ∝ $\sum_{s\in S}$ [(# edges within group *s*) – (expected # edges within group *s*)]**
  - $$Q(G,S) = \underbrace{\frac{1}{2m}}_{\text{Normalizing cost.: } -1<Q<1} \sum_{s\in S} \sum_{i\in s} \sum_{j\in s} \left(A_{ij} - \frac{k_i k_j}{2m}\right)$$

A_{ij} = 1 if i→j,
0 else

- **Modularity values take range [–1,1]**
  - It is positive if the number of edges within groups exceeds the expected number
  - **0.3-0.7<Q** means significant community structure

Modularity: Number of clusters

- **Modularity is useful for selecting the number of clusters:**

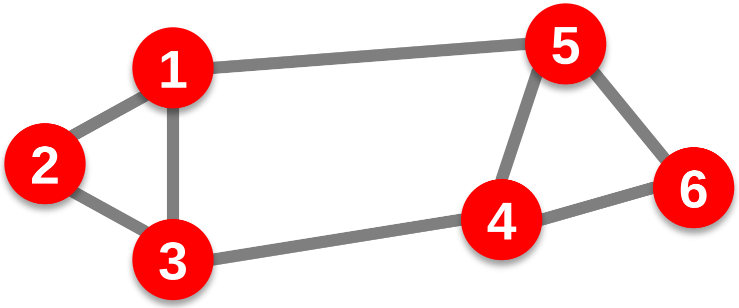


Next time: Why not optimize Modularity directly?

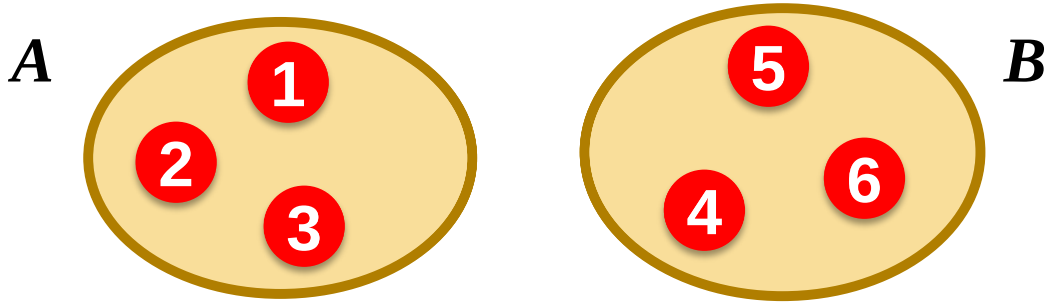
Spectral Clustering

Graph Partitioning

- **Undirected graph $G(V, E)$:**



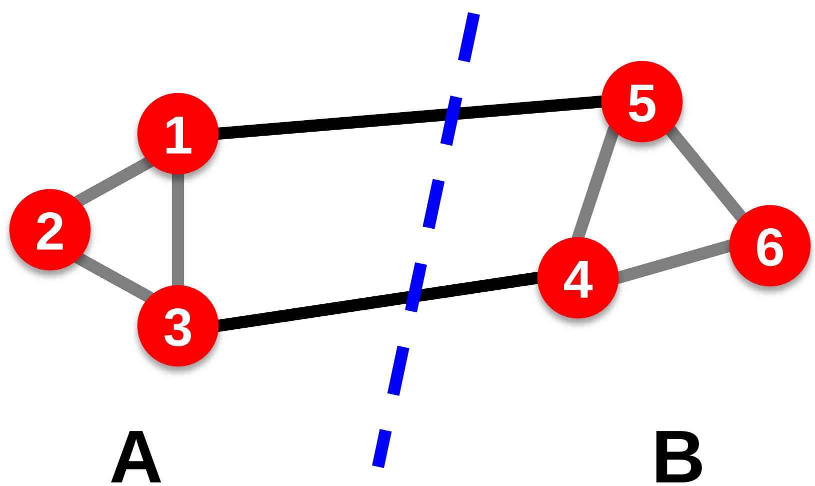
- **Bi-partitioning task:**
  - Divide vertices into two disjoint groups A, B



- **Questions:**
  - How can we define a "good" partition of G?
  - How can we efficiently identify such a partition?

Graph Partitioning

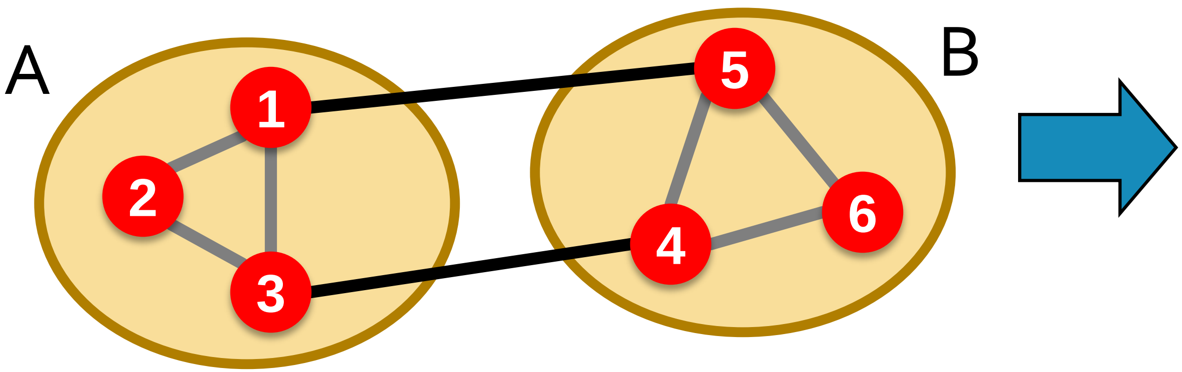
- **What makes a good partition?**
  - Maximize the number of within-group connections
  - Minimize the number of between-group connections



Graph Cuts

- **Express partitioning objectives as a function of the "edge cut" of the partition**
- **Cut:** Set of edges with only one vertex in a group:

$$cut(A,B) = \sum_{i \in A, j \in B} w_{ij}$$



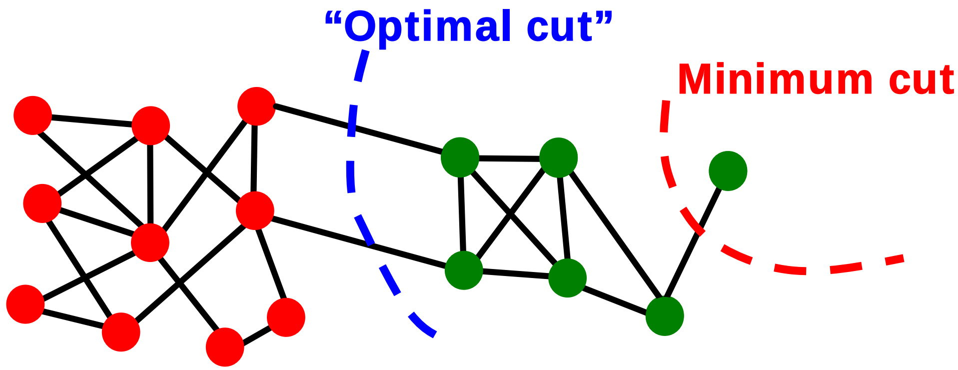
$cut(A,B) = 2$

Graph Cut Criterion

- **Criterion: Minimum-cut**
  - Minimize weight of connections between groups

$$\arg\min_{A,B} cut(A,B)$$

- **Degenerate case:**



- **Problem:**
  - Only considers external cluster connections
  - Does not consider internal cluster connectivity

Graph Cut Criteria

- **Criterion: Normalized-cut** [Shi-Malik, '97]
  - Connectivity between groups relative to the density of each group

$$ncut(A,B) = \frac{cut(A,B)}{vol(A)} + \frac{cut(A,B)}{vol(B)}$$

$\boldsymbol{vol(A)}$: total weight of the edges with at least one endpoint in $\boldsymbol{A}$: $\boldsymbol{vol(A) = \sum_{i \in A} k_i}$

- **Why use this criterion?**
  - Produces more balanced partitions
- **How do we efficiently find a good partition?**
  - **Problem:** Computing optimal cut is NP-hard

Spectral Graph Partitioning

- A: adjacency matrix of undirected **G**
  - A_{ij} **=1** if $(\boldsymbol{i}, \boldsymbol{j})$ is an edge, else **0**
- $\boldsymbol{x}$ is a vector in $\Re^n$ with components $(\boldsymbol{x_1}, \dots, \boldsymbol{x_n})$
  - Think of it as a label/value of each node of $\boldsymbol{G}$
- **What is the meaning of $A \cdot x$?**

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$y_i = \sum_{j=1}^{n} A_{ij} xj = \sum_{(i,j)\in E} x_j$$

- **Entry y_i is a sum of labels x_j of neighbors of i**

What is the meaning of Ax?

- j^{th} **coordinate of** $A \cdot x$:
  - Sum of the x-values of neighbors of j
  - Make this a new value at node j

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{A \cdot x = \lambda \cdot x}$$

- **Spectral Graph Theory:**
  - Analyze the "spectrum" of matrix representing G
  - **Spectrum:** Eigenvectors x_i of a graph, ordered by the magnitude (strength) of their corresponding eigenvalues λ_i: $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

Example: d-regular graph

- Suppose all nodes in $\boldsymbol{G}$ have degree $\boldsymbol{d}$ and $\boldsymbol{G}$ is connected
- **What are some eigenvalues/vectors of $\boldsymbol{G}$?**

$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{\lambda} \cdot \boldsymbol{x}$ What is λ? What x?

  - **Let's try:** $\boldsymbol{x} = (1, 1, \dots, 1)$
  - **Then:** $\boldsymbol{A} \cdot \boldsymbol{x} = (\boldsymbol{d}, \boldsymbol{d}, \dots, \boldsymbol{d}) = \boldsymbol{\lambda} \cdot \boldsymbol{x}$. **So:** $\boldsymbol{\lambda} = \boldsymbol{d}$
  - **We found eigenpair of $\boldsymbol{G}$:** $\boldsymbol{x} = (1, 1, \dots, 1), \boldsymbol{\lambda} = \boldsymbol{d}$

Remember the meaning of $\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{x}$:

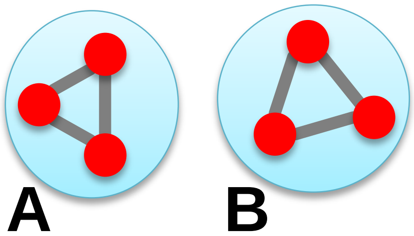
$$y_j = \sum_{i=1}^{n} A_{ij} x_i = \sum_{(j,i) \in E} x_i$$

d is the largest eigenvalue of *A*

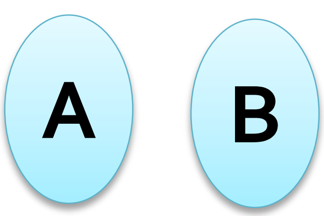
Details!

- **G** is **d**-regular connected, **A** is its adjacency matrix
- **Claim:**
  - **d** is largest eigenvalue of **A**,
  - **d** has multiplicity of **1** (there is only **1** eigenvector associated with eigenvalue **d**)
- **Proof: Why no eigenvalue $d' > d$?**
  - To obtain **d** we needed $x_i = x_j$ for every i, j
  - This means $x = c \cdot (1,1, \dots, 1)$ for some const. c
  - **Define:** S = nodes i with maximum possible value of x_i
  - Then consider some vector y which is not a multiple of vector $(\mathbf{1}, \dots, \mathbf{1})$. So not all nodes i (with labels y_i) are in S
  - Consider some node $j \in S$ and a neighbor $i \notin S$ then node j gets a value strictly less than d
  - So y is not eigenvector! And so d is the largest eigenvalue!

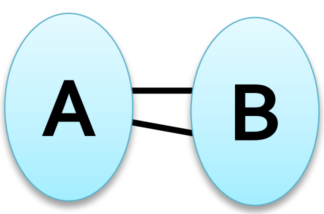
Example: Graph on 2 components



- **What if G is not connected?**
  - G has **2** components, each d-regular
- **What are some eigenvectors?**
  - $x =$ Put all $\mathbf{1}$s on A and $\mathbf{0}$s on B or vice versa
    - $x' = (\underbrace{1, \dots, 1}_{|A|}, \underbrace{0, \dots, 0}_{|B|})$ **then** $\mathbf{A} \cdot x' = (d, \dots, d, 0, \dots, 0)$
    - $x'' = (0, \dots, 0, 1, \dots, 1)$ **then** $A \cdot x'' = (0, \dots, 0, d, \dots, d)$
    - And so in both cases the corresponding $\lambda = d$
- **A bit of intuition:**



$\lambda_n = \lambda_{n-1}$

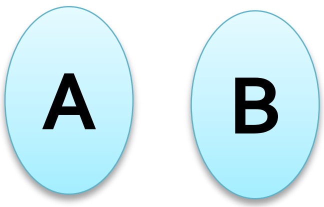


$\lambda_n - \lambda_{n-1} \approx 0$

2nd largest eigval. λ_{n-1} now has value very close to λ_n

More Intuition

- **More intuition:**



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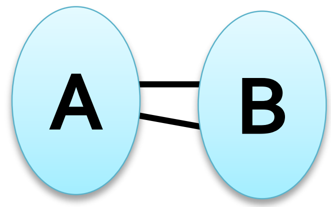


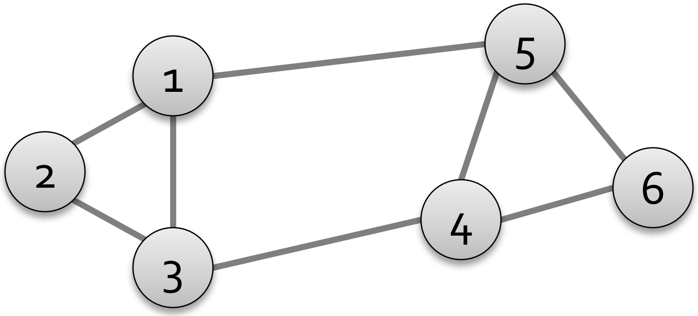
$\lambda_n - \lambda_{n-1} \approx 0$

2nd largest eigval. λ_{n-1} now has value very close to λ_n

- If the graph is connected (right example) then we already know that $\boldsymbol{x_n} = (\mathbf{1}, \dots \mathbf{1})$ is an eigenvector
- Since eigenvectors are orthogonal then the components of $\boldsymbol{x_{n-1}}$ sum to **0**.
  - Why? Because $\boldsymbol{x_n} \cdot \boldsymbol{x_{n-1}} = \sum_i \boldsymbol{x_n}[\boldsymbol{i}] \cdot \boldsymbol{x_{n-1}}[\boldsymbol{i}]$
- So we can look at the eigenvector of the 2nd largest eigenvalue and declare nodes with positive label in **A** and negative label in **B.**
- **But there is still lots to sort out.**

Matrix Representations

- **Adjacency matrix (A):**
  - $n \times n$ matrix
  - $A=[a_{ij}]$, $a_{ij}=1$ if edge between node i and j

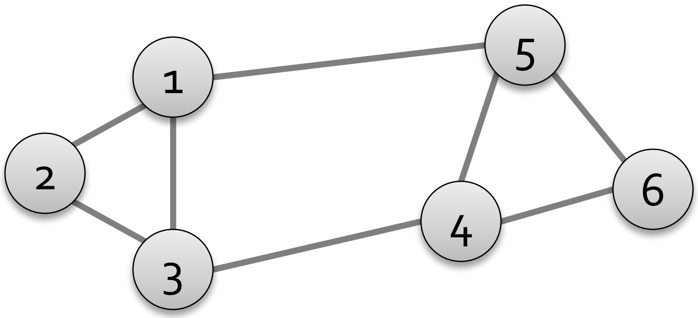


	1	2	3	4	5	6
1	0	1	1	0	1	0
2	1	0	1	0	0	0
3	1	1	0	1	0	0
4	0	0	1	0	1	1
5	1	0	0	1	0	1
6	0	0	0	1	1	0

- **Important properties:**
  - Symmetric matrix
  - Eigenvectors are real and orthogonal

Matrix Representations

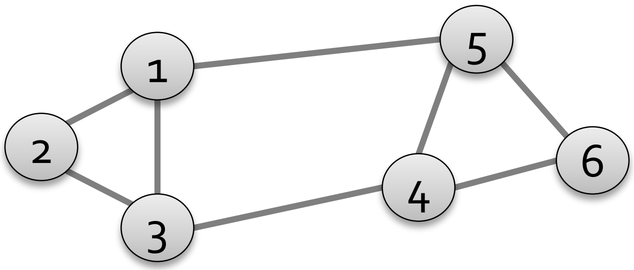
- **Degree matrix (D):**
  - $n \times n$ diagonal matrix
  - $D=[d_{ii}]$, d_{ii} = degree of node i



	1	2	3	4	5	6
1	3	0	0	0	0	0
2	0	2	0	0	0	0
3	0	0	3	0	0	0
4	0	0	0	3	0	0
5	0	0	0	0	3	0
6	0	0	0	0	0	2

Matrix Representations

- **Laplacian matrix (L):**
  - $n \times n$ symmetric matrix



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

$$\boldsymbol{L = D - A}$$

- **What is trivial eigenpair?**
  - $\boldsymbol{x} = (1, \dots, 1)$ then $\boldsymbol{L \cdot x = 0}$ and so $\boldsymbol{\lambda = \lambda_1 = 0}$
- **Important properties:**
  - **Eigenvalues** are non-negative real numbers
  - **Eigenvectors** are real and orthogonal

Facts about the Laplacian L

Details!

(a) All eigenvalues are ≥ 0

(b) $x^T L x = \sum_{ij} L_{ij} x_i x_j \geq 0$ for every x

(c) $L = N^T \cdot N$

- That is, L is positive semi-definite

- **Proof:**
  - **(c)⇒(b):** $x^T L x = x^T N^T N x = (xN)^T (Nx) \geq 0$
    - As it is just the square of length of Nx
  - **(b)⇒(a):** Let $\boldsymbol{\lambda}$ be an eigenvalue of $\boldsymbol{L}$. Then by **(b)** $x^T L x \geq 0$ so $x^T L x = x^T \lambda x = \lambda x^T x \Rightarrow \boldsymbol{\lambda \geq 0}$
  - **(a)⇒(c):** is also easy! Do it yourself.

λ_2 as optimization problem

- **Fact: For symmetric matrix M:**

$$\lambda_2 = \min_x \frac{x^T M\, x}{x^T x}$$

- **What is the meaning of min $x^T L\, x$ on G?**
  - $\mathrm{x}^{\mathrm{T}}\mathrm{L}\,\mathrm{x} = \sum_{i,j=1}^{n} L_{ij}\, x_i x_j = \sum_{i,j=1}^{n} \left(D_{ij} - A_{ij}\right) x_i x_j$
  - $= \sum_i D_{ii} x_i^2 - \sum_{(i,j)\in E} 2 x_i x_j$
  - $= \sum_{(i,j)\in E} (\underbrace{x_i^2 + x_j^2} - 2x_i x_j) = \sum_{(i,j)\in E} \left(x_i - x_j\right)^2$

Node i has degree d_i. So, value x_i^2 needs to be summed up d_i times.
But each edge (i,j) has two endpoints so we need $x_i^2 + x_j^2$

Proof: $\lambda_2 = \min_x \frac{x^T M\ x}{x^T x}$

Details!

- Write x in axes of eigenvecotrs $w_1, w_2, \dots, w_n$ of $\boldsymbol{M}$. So, $x = \sum_i^n \alpha_i w_i$
- Then we get: $Mx = \sum_i \alpha_i \underbrace{M w_i}_{\lambda_i w_i} = \sum_i \alpha_i \lambda_i w_i$
- So, what is $\boldsymbol{x^T M x}$?
  - $x^T M x = (\sum_i \alpha_i w_i)(\sum_i \alpha_i \lambda_i w_i) = \sum_{ij} \alpha_i \lambda_j \alpha_j \overbrace{w_i w_j}^{\substack{= 0 \text{ if } i \neq j \\ 1 \text{ otherwise}}}$

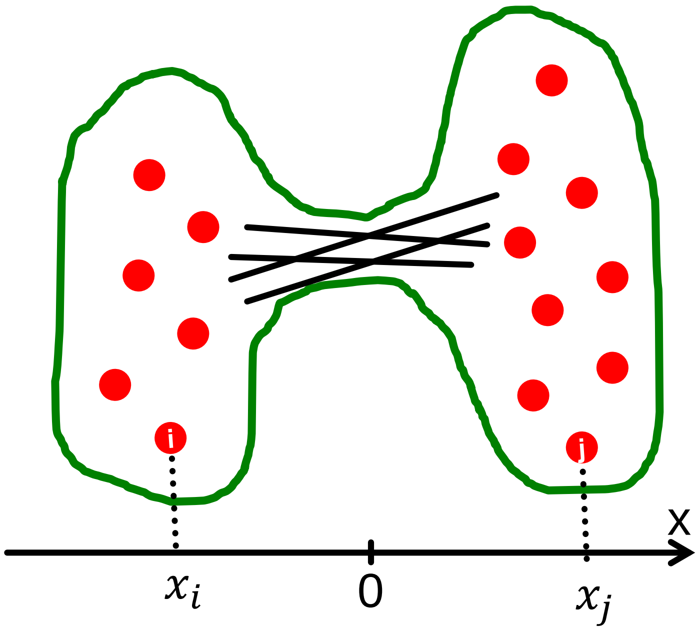
    $= \sum_i \alpha_i \lambda_i w_i w_i = \boldsymbol{\sum_i \lambda_i \alpha_i^2}$
  - To minimize this over all unit vectors x orthogonal to:
    w = min over choices of $(\alpha_1, \dots \alpha_n)$ so that:
    $\sum \alpha_i^2 = 1$ (unit length) $\sum \alpha_i = 0$ (orthogonal to w_1)
  - To minimize this, set $\boldsymbol{\alpha_2 = 1}$ and so $\boldsymbol{\sum_i \lambda_i \alpha_i^2 = \lambda_2}$

λ_2 as optimization problem

- **What else do we know about x?**
  - $\boldsymbol{x}$ is unit vector: $\sum_i x_i^2 = 1$
  - $\boldsymbol{x}$ is orthogonal to **1**st eigenvector $(1, \dots, 1)$ thus: $\sum_i x_i \cdot 1 = \sum_i x_i = 0$
- **Remember:**

$$\lambda_2 = \min_{\substack{\text{All labelings} \\ \text{of nodes } i \text{ so} \\ \text{that } \sum x_i = 0}} \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_i x_i^2}$$

We want to assign values x_i to nodes i such that few edges cross 0.
(we want x_i and x_j to subtract each other)



Balance to minimize

Find Optimal Cut [Fiedler'73]

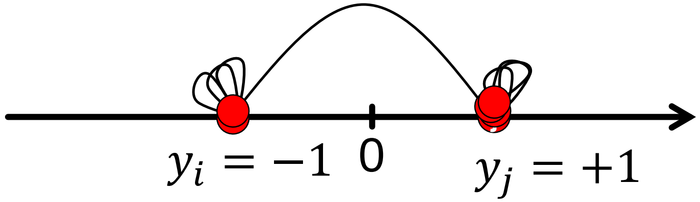
- **Back to finding the optimal cut**
- **Express partition (A,B) as a vector**

$$y_i = \begin{cases} +1 & if\ i \in A \\ -1 & if\ i \in B \end{cases}$$

- We can minimize the cut of the partition by finding a non-trivial vector x that **minimizes**:

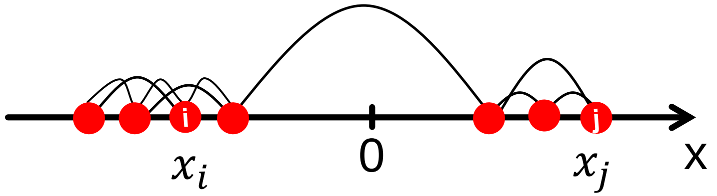
$$\arg\min_{y \in [-1,+1]^n} f(y) = \sum_{(i,j)\in E} (y_i - y_j)^2$$

Can't solve exactly. Let's relax y and allow it to take any real value.



Rayleigh Theorem

$$\min_{y \in \Re^n} f(y) = \sum_{(i,j) \in E} (y_i - y_j)^2 = y^T L y$$



- $\lambda_2 = \min_y f(y)$: The minimum value of $f(y)$ is given by the 2nd smallest eigenvalue λ_2 of the Laplacian matrix L
- $\mathbf{x} = \arg\min_{\mathbf{y}} f(y)$: The optimal solution for y is given by the corresponding eigenvector x, referred as the **Fiedler vector**

Approx. Guarantee of Spectral

Details!

- Suppose there is a partition of **G** into **A** and **B** where $|A| \leq |B|$, s.t. $\boldsymbol{\alpha} = \frac{(\#\ edges\ from\ A\ to\ B)}{|A|}$ then $2\boldsymbol{\alpha} \geq \boldsymbol{\lambda_2}$
  - This is the approximation guarantee of the spectral clustering. It says the cut spectral finds is at most **2** away from the optimal one of score $\boldsymbol{\alpha}$.
- **Proof:**
  - Let: **a=|A|, b=|B|** and **e=** # edges from **A** to **B**
  - Enough to choose some $\boldsymbol{x_i}$ based on **A** and **B** such that: $\lambda_2 \leq \underbrace{\frac{\sum (x_i - x_j)^2}{\sum_i x_i^2}}_{\lambda_2 \text{ is only smaller}} \leq 2\alpha$ (while also $\sum_i x_i = 0$)

Approx. Guarantee of Spectral

Details!

- **Proof (continued):**
  - **1) Let's set:** $\boldsymbol{x_i} = \begin{cases} -\frac{\mathbf{1}}{\boldsymbol{a}} & \boldsymbol{if\ i \in A} \\ +\frac{\mathbf{1}}{\boldsymbol{b}} & \boldsymbol{if\ i \in B} \end{cases}$
    - Let's quickly verify that $\sum_i x_i = 0$: $a\left(-\frac{1}{a}\right) + b\left(\frac{1}{b}\right) = \mathbf{0}$
  - **2) Then:** $\frac{\sum (x_i - x_j)^2}{\sum_i x_i^2} = \frac{\sum_{i \in A, j \in B}\left(\frac{1}{b}+\frac{1}{a}\right)^2}{a\left(-\frac{1}{a}\right)^2 + b\left(\frac{1}{b}\right)^2} = \frac{e \cdot \left(\frac{1}{a}+\frac{1}{b}\right)^2}{\frac{1}{a}+\frac{1}{b}} =$

$$e\left(\frac{1}{a}+\frac{1}{b}\right) \leq e\left(\frac{1}{a}+\frac{1}{a}\right) \leq \boldsymbol{e}\frac{\mathbf{2}}{\boldsymbol{a}} = \mathbf{2}\boldsymbol{\alpha}$$

e … number of edges between A and B

Which proves that the cost achieved by spectral is better than twice the OPT cost

Approx. Guarantee of Spectral

Details!

- **Putting it all together:**

$$2\alpha \geq \lambda_2 \geq \frac{\alpha^2}{2k_{max}}$$

  - where k_{max} is the maximum node degree in the graph
    - Note we only provide the 1st part: $2\alpha \geq \lambda_2$
    - We did not prove $\lambda_2 \geq \frac{\alpha^2}{2k_{max}}$
  - Overall this always certifies that λ_2 always gives a useful bound

So far…

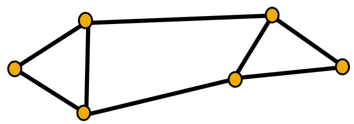
- **How to define a "good" partition of a graph?**
  - Minimize a given graph cut criterion
- **How to efficiently identify such a partition?**
  - Approximate using information provided by the eigenvalues and eigenvectors of a graph
- **Spectral Clustering**

Spectral Clustering Algorithms

- **Three basic stages:**
  - **1) Pre-processing**
    - Construct a matrix representation of the graph
  - **2) Decomposition**
    - Compute eigenvalues and eigenvectors of the matrix
    - Map each point to a lower-dimensional representation based on one or more eigenvectors
  - **3) Grouping**
    - Assign points to two or more clusters, based on the new representation

Spectral Partitioning Algorithm

- **1) Pre-processing:**
  - Build Laplacian matrix L of the graph



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

- **2) Decomposition:**
  - Find eigenvalues λ and eigenvectors x of the matrix L
  - Map vertices to corresponding components of λ_2

λ=

0.0
1.0
3.0
3.0
4.0
5.0

X =

0.4	0.3	-0.5	-0.2	-0.4	-0.5
0.4	0.6	0.4	-0.4	0.4	0.0
0.4	0.3	0.1	0.6	-0.4	0.5
0.4	-0.3	0.1	0.6	0.4	-0.5
0.4	-0.3	-0.5	-0.2	0.4	0.5
0.4	-0.6	0.4	-0.4	-0.4	0.0

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

How do we now find the clusters?

Spectral Partitioning

- 3) **Grouping:**
  - Sort components of reduced 1-dimensional vector
  - Identify clusters by splitting the sorted vector in two
- **How to choose a splitting point?**
  - Naïve approaches:
    - Split at **0** or median value
  - More expensive approaches:
    - Attempt to minimize normalized cut in 1-dimension (sweep over ordering of nodes induced by the eigenvector)

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

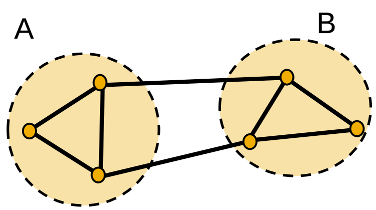
Split at 0:

Cluster A: Positive points

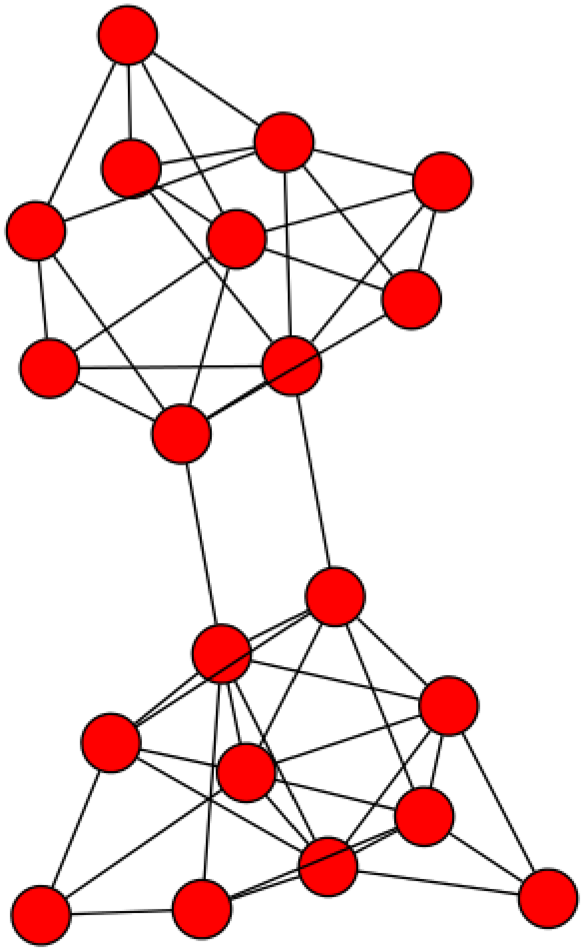
Cluster B: Negative points

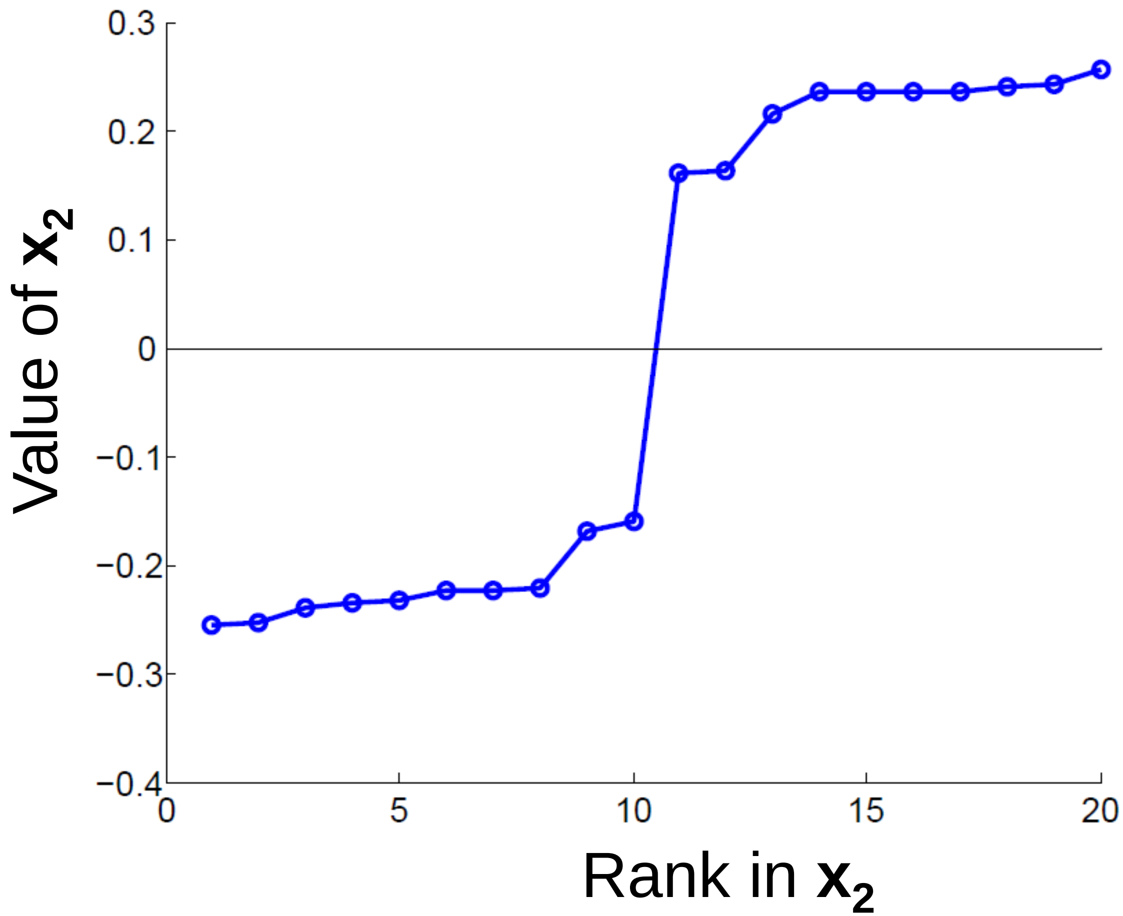
1	0.3
2	0.6
3	0.3

4	-0.3
5	-0.3
6	-0.6

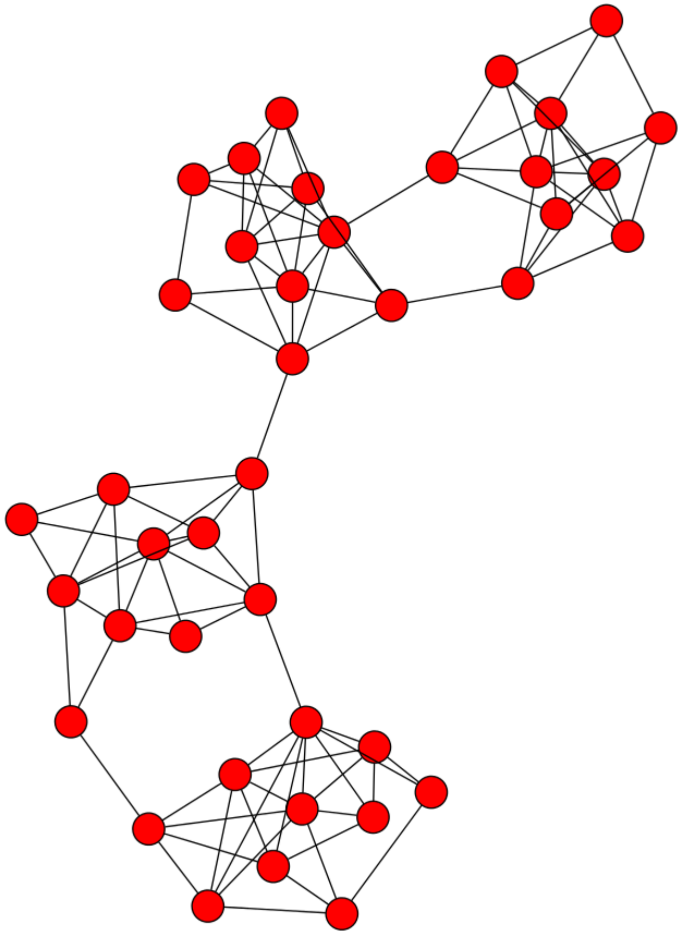


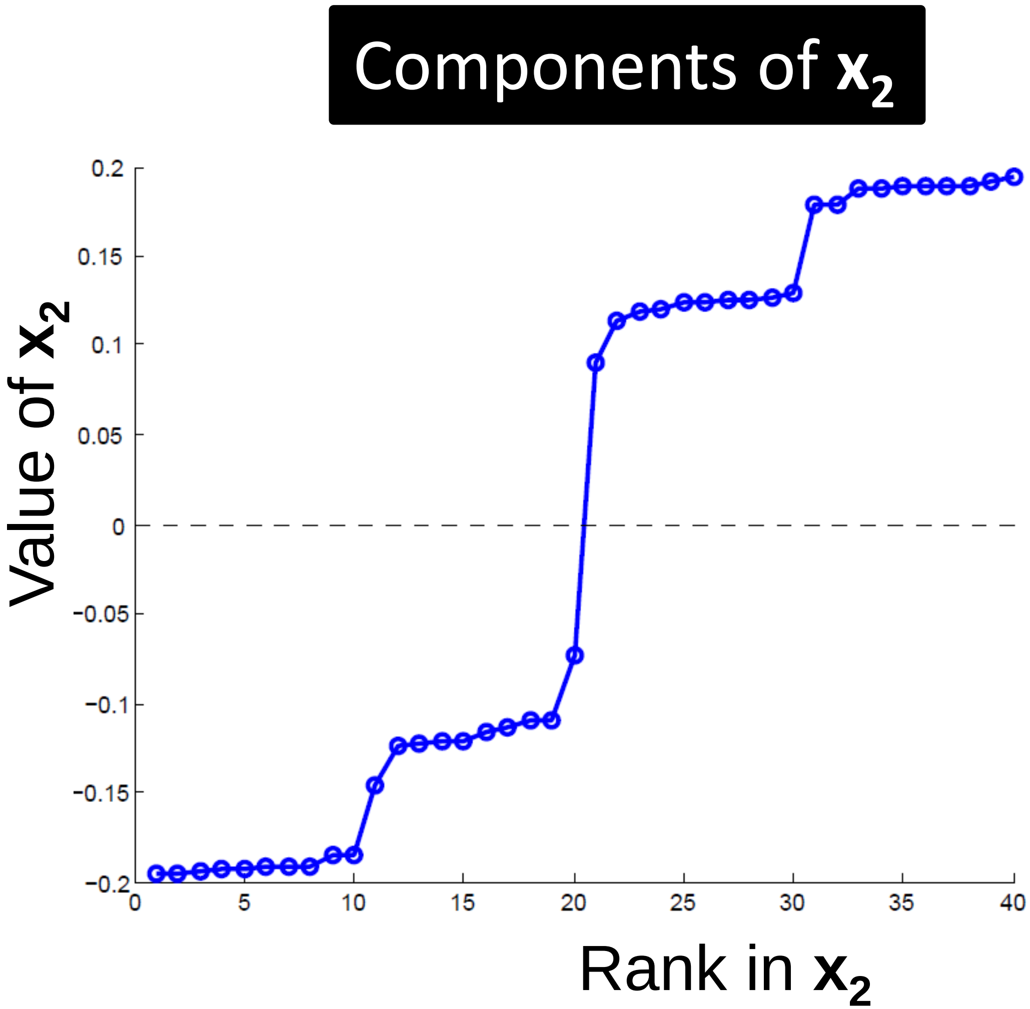
Example: Spectral Partitioning



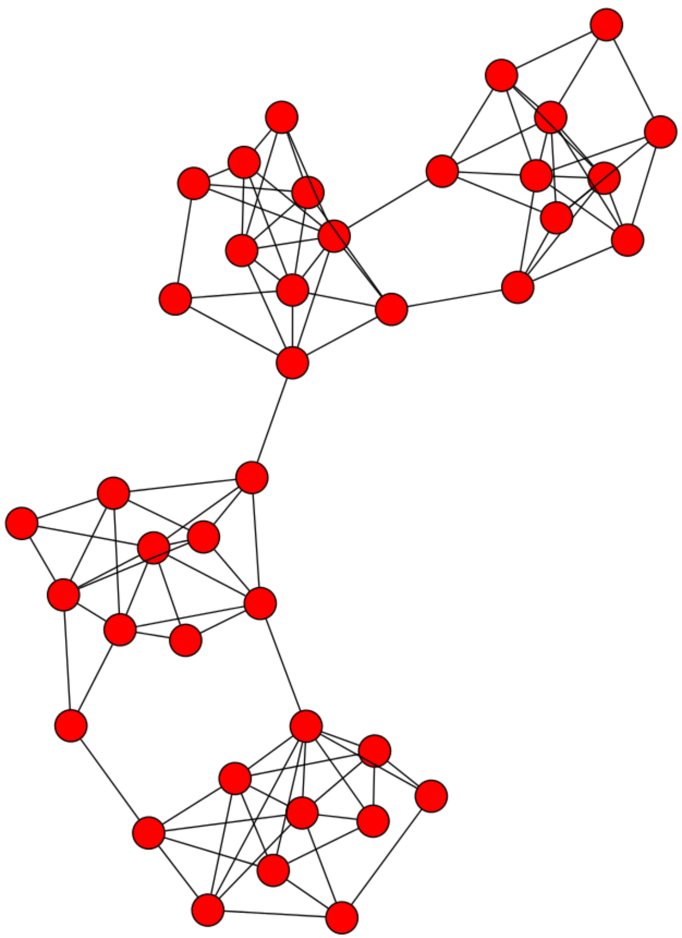


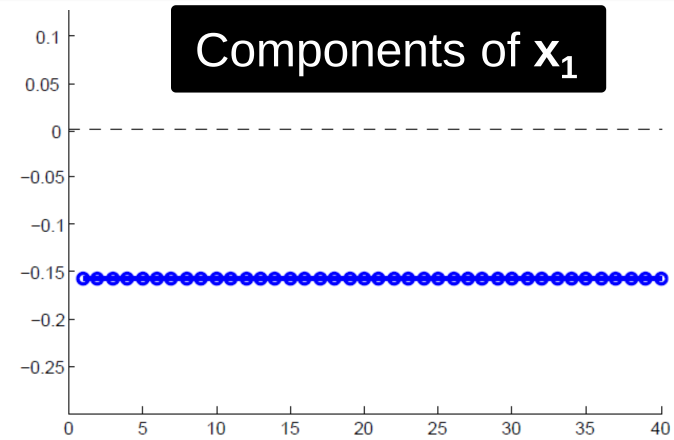
Example: Spectral Partitioning

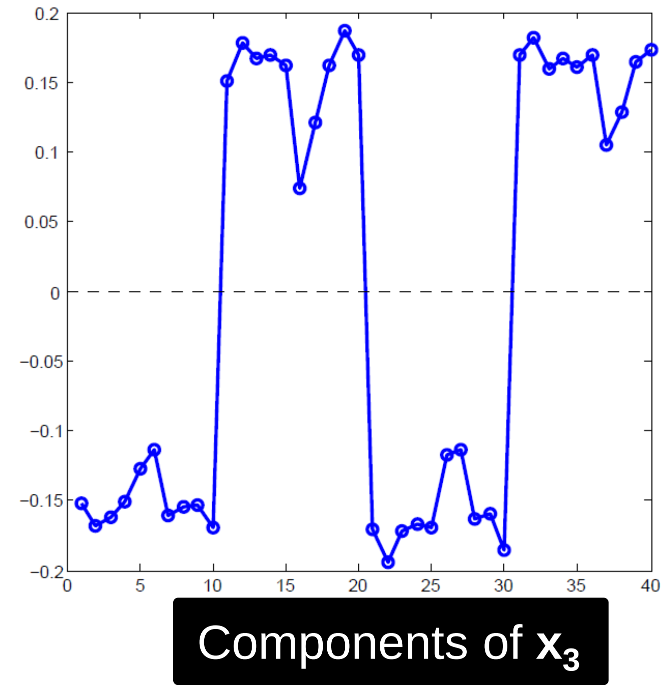




Example: Spectral partitioning







k-Way Spectral Clustering

- **How do we partition a graph into *k* clusters?**
- **Two basic approaches:**
  - **Recursive bi-partitioning** [Hagen et al., ’92]
    - Recursively apply bi-partitioning algorithm in a hierarchical divisive manner
    - Disadvantages: Inefficient, unstable
  - **Cluster multiple eigenvectors** [Shi-Malik, ’00]
    - Build a reduced space from multiple eigenvectors
    - Commonly used in recent papers
    - A preferable approach…

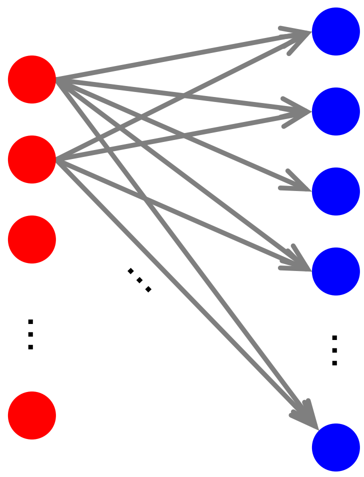
Why use multiple eigenvectors?

- **Approximates the optimal cut** [Shi-Malik, '00]
  - Can be used to approximate optimal k-way normalized cut
- **Emphasizes cohesive clusters**
  - Increases the unevenness in the distribution of the data
  - Associations between similar points are amplified, associations between dissimilar points are attenuated
  - The data begins to "approximate a clustering"
- **Well-separated space**
  - Transforms data to a new "embedded space", consisting of k orthogonal basis vectors
- Multiple eigenvectors prevent instability due to information loss

Analysis of Large Graphs: Trawling

Trawling

- **Searching for small communities in the Web graph**
- **What is the signature of a community / discussion in a Web graph?**



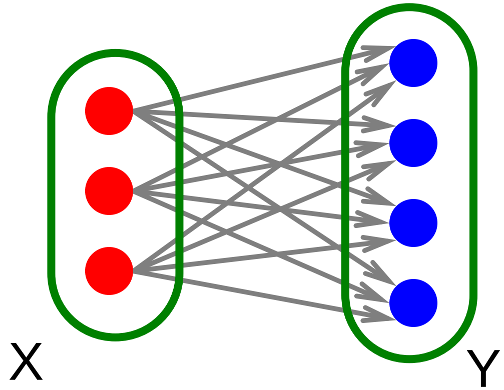
Dense 2-layer graph

Use this to define "topics":
What the same people on the left talk about on the right
Remember HITS!

Intuition: Many people all talking about the same things

Searching for Small Communities

- **A more well-defined problem:**
  Enumerate complete bipartite subgraphs $K_{s,t}$
  - Where $K_{s,t}$: s nodes on the "left" where each links to the same t other nodes on the "right"



$K_{3,4}$

$|X| = s = 3$
$|Y| = t = 4$

Fully connected

Frequent Itemset Enumeration

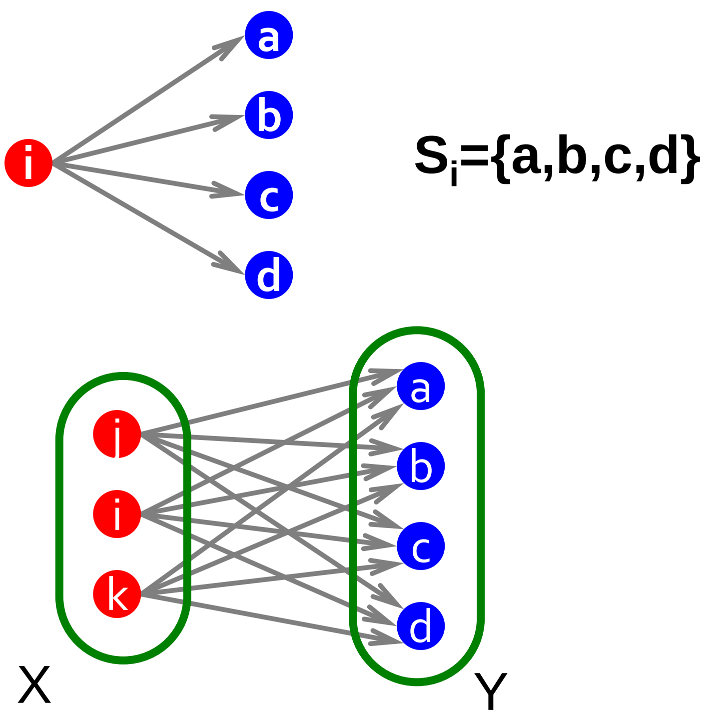
- **Market basket analysis.** Setting:
  - **Market:** Universe U of n items
  - **Baskets:** m subsets of U: $S_1, S_2, \ldots, S_m \subseteq U$ (S_i is a set of items one person bought)
  - **Support:** Frequency threshold f
- **Goal:**
  - **Find all subsets T s.t. $T \subseteq S_i$ of at least f sets S_i** (items in T were bought together at least f times)
- **What's the connection between the itemsets and complete bipartite graphs?**

[Kumar et al. '99]

From Itemsets to Bipartite $K_{s,t}$

Frequent itemsets = complete bipartite graphs!

- **How?**
  - View each node i as a set S_i of nodes i points to
  - $K_{s,t}$ = a set Y of size t that occurs in s sets S_i
  - Looking for $K_{s,t}$ → set of frequency threshold to s and look at layer t – all frequent sets of size t

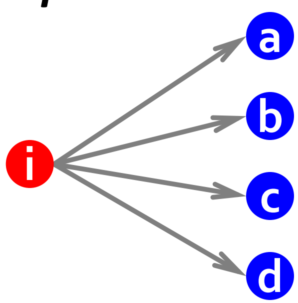


s … minimum support (|X|=s)
t … itemset size (|Y|=t)

[Kumar et al. '99]

From Itemsets to Bipartite $K_{s,t}$

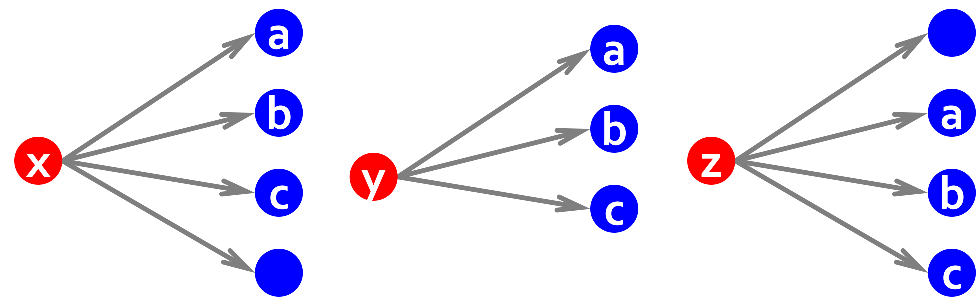
View each node ***i*** as a set ***S_i*** of nodes ***i*** points to



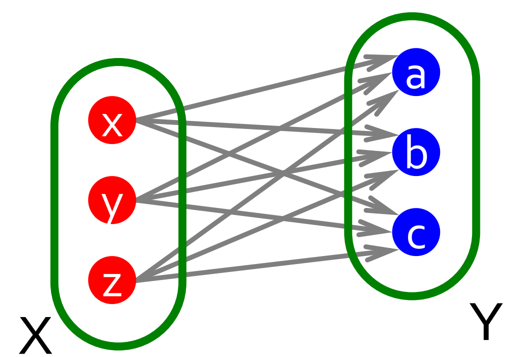
S_i={a,b,c,d}

Find frequent itemsets:
- **s** … minimum support
- **t** … itemset size

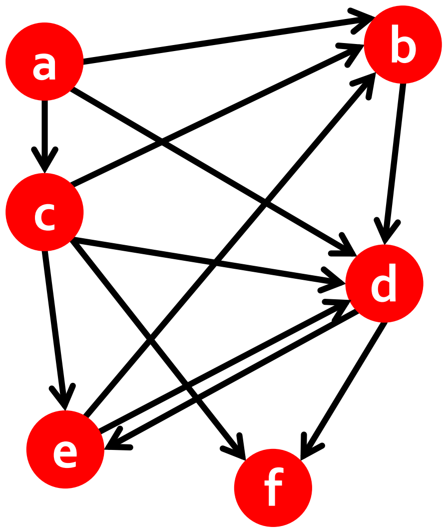
Say we find a **frequent itemset *Y={a,b,c}*** of supp ***s***
So, there are ***s*** nodes that link to all of **{a,b,c}**:



We found *$K_{s,t}$*!
$K_{s,t}$ = a set ***Y*** of size ***t*** that occurs in *s* sets ***S_i***



Example (1)



Itemsets:

a = {b,c,d}

b = {d}

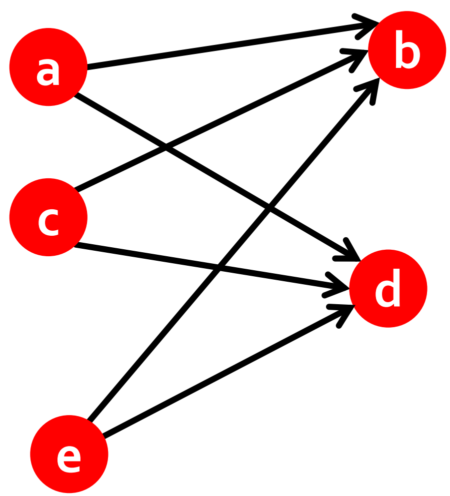
c = {b,d,e,f}

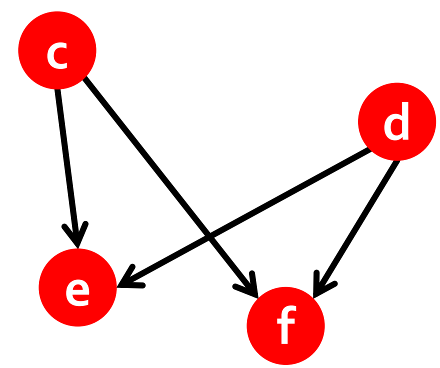
d = {e,f}

e = {b,d}

f = {}

- **Support threshold s=2**
  - **{b,d}**: support 3
  - **{e,f}**: support 2
- **And we just found 2 bipartite subgraphs:**





Example (2)

- **Example of a community from a web graph**

A community of Australian fire brigades

Nodes on the right	Nodes on the left
NSW Rural Fire Service Internet Site	New South Wales Fir...ial Australian Links
NSW Fire Brigades	Feuerwehrlinks Australien
Sutherland Rural Fire Service	FireNet Information Network
CFA: County Fire Authority	The Cherrybrook Rur...re Brigade Home Page
"The National Cente...ted Children's Ho...	New South Wales Fir...ial Australian Links
CRAFTI Internet Connexions-INFO	Fire Departments, F... Information Network
Welcome to Blackwoo... Fire Safety Serv...	The Australian Firefighter Page
The World Famous Guestbook Server	Kristiansand brannv...dens brannvesener...
Wilberforce County Fire Brigade	Australian Fire Services Links
NEW SOUTH WALES FIR...ES 377 STATION	The 911 F,P,M., Fir...mp; Canada A Section
Woronora Bushfire Brigade	Feuerwehrlinks Australien
Mongarlowe Bush Fire – Home Page	Sanctuary Point Rural Fire Brigade
Golden Square Fire Brigade	Fire Trails "l...ghters around the...
FIREBREAK Home Page	FireSafe – Fire and Safety Directory
Guises Creek Volunt...fficial Home Page...	Kristiansand Firede...departments of th...

[Kumar, Raghavan, Rajagopalan, Tomkins: Trawling the Web for emerging cyber-communities 1999]