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OVERCOMING THE OBSTACLES IN FUNCTION UNDERSTANDING WITH THE CONTRIBUTION OF TECHNOLOGY BASED ON MODELLING ACTIVITIES IN HIGH SCHOOL PHYSICS

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Abstract. Understanding concepts and the relations among them is greatly facilitated by the modelling process. We wanted our students to achieve the development of qualitative and semi-quantitave premise and then express magnitude relationships in a quantitative manner via the use of modelling activities. This paper focuses on overcoming some obfuscations of the function understanding with the contribution of modelling activities based on electronic environment.

Zusammenfassung. Zusammenfassung. Wir meinen, dass das Verständnis von Begriffen und deren Zusammenhänge, besonders durch die Erstellung von Modellen erleichtert wird. Durch die Aktivitäten der Modellerstellung in der Kinematik, haben wir versucht, dass die Schüler das qualitative und semiquantitative Denken entwickeln und dass sie zum quantitativen Ausdruck der Grössenverhältnisse übergehen. In diesem Artikel wird vorgestellt wie Hindernisse des Begreifens von Verbindungen mit Hilfe von Aktivitäten der Modellersstellung überwunden werden, so dass die Schüler die Verbindungen in de Physik von Grund auf verstehen können.

Résumé. Nous considérons que la compréhension des significations et les relations entre eux, est facilitée dans une grande mesure par la procédure de modélisation. Avec les activités de modélisation, en edudiant les mouvements, nous avons essayé les élèves qu'ils développent le raisonnement qualitatif et semiquantitative qu'ils passent à l'expression quantitative de la relation des tailles. Au présent article il se présente qu'ont été surmontés des obstacles de compréhension des relations avec la contribution des activités de modélisation afin que les élèves comprennent en profondeur.

Riassunto. Noi crediamo che la comprensione dei concetti e le relazioni tra di loro sono enormemente facilitate dal processo di modellazione. Nelle attività di modellazione in cinematica é cercato che agli studenti si sviluppase il ragionamento qualitativo e semi-quantitativo in modo che loro potrebbero pasare alla espressione quantitativa del rapporto tra le dimensioni. Questo articolo mostra come sono stati superati ostacoli di

compressione delle funzioni con il contributo delle attività di modellazione in modo che gli studenti possano comprendere in profondità le funzioni nel campo della fisica.

Περίληψη. Θεωρούμε ότι η κατανόηση των εννοιών και οι σχέσεις μεταξύ των διευκολύνεται σε μεγάλο βαθμό από τη διαδικασία μοντελοποίησης. Με τις δραστηριότητες μοντελοποίησης στην Κινηματική προσπαθήσαμε οι μαθητές να αναπτύξουν τον ποιοτικό και ημιποσοτικό συλλογισμό και να περάσουν στην ποσοτική έκφραση της σχέσης των μεγεθών. Στο παρόν άρθρο παρουσιάζεται πως υπερνικήθηκαν εμπόδια κατανόησης των συναρτήσεων με τη συμβολή των δραστηριοτήτων μοντελοποίησης, με έμφαση στη Φυσική.

Key words: modelling process, representations, students' activities, scaffolding, function, kinematical concepts, qualitative reasoning.

1 INTRODUCTION

According to Goudas & Sakonidis (2002), the concept of function and in particular the manipulation of its formal definition and its representations constitute a difficult area to comprehend and handle for many children. The concept of function remains a difficult mathematical concept for most students, despite several efforts. In literature, three main parameters of this difficulty are mentioned. The first is the complexity of the concept and the variety of mathematical meanings associated with it, such as variable, plus-change, and all others. The second relates to the fact that the concept of function is inherent in a great part of mathematics and school mathematics: the four operations, measurement geometry, solving equations and other techniques and algorithms can be studied in terms of functions. This is particularly difficult to create a unified and generally accepted framework in order to learn the meaning of the function. The third factor relates to the need that students should understand the meaning of the function at a level as a process and another level as an object. For example, in order the student to interpret the graph of a function, he needs to understand the function as a process, but in order to study the secondary or the integration he should understand the three components of the concept, i.e. the range, the domain and the matching rule as a single conceptual entity (Harel & Kaput, 1991). The current situation of school mathematics doesn't give students the opportunity to realize and exploit this distinction. On the contrary, usually the perception of one concept of function is emphasized over the other, with serious consequences in the knowledge obtained by students. Their research indicates that although most pupils understand a function as a computational process, they find it difficult to relate the algebraic with its graphical representation.

It is not uncommon to encounter students who are able to solve problems by using complicated relations without getting a qualitative and in-depth grip on them. Researchers have already shown that students often have a formal mathematical and physical knowledge without a qualitative understanding of basic concepts and relations Jimoyiannis & Komis (2001), Niedderer (1991), Smyrnaiou & Weil-Barais (2003).

In sciences, 'concepts' work as structural elements of the cognitive edifice. In theory, a concept relates to others via axioms, definitions and laws; the network of which constitutes the organisation of the concept. In the students' minds, the concepts are feebly structured and partial. However, a concept conceived in isolation is practically without meaning. Most students think of Physics as a collection of facts that have to be memorized. This tendency weakens their ability to discern the beautiful structure of the natural world that science reveals to us Hestenes (1992).

There is a basic difference between mathematics and other domains of scientific knowledge as the only way to access mathematical objects and deal with them is by using signs and semiotic representations. Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2006).

The use of multiple representations has been strongly connected with the complex process of learning in mathematics, and more particularly, with the seeking of students' better understanding of important mathematical concepts (Greeno & Hall, 1997), such as function. Mathematics instructors, at the secondary level, traditionally have focused their teaching on the use of the algebraic representation of functions (Eisenberg & Dreyfus, 1991). Sfard (1992) showed that students were unable to bridge the algebraic and graphical representations of functions, while Markovits, Eylon and Bruckheimer (1986) observed that the translation from graphical to algebraic form was more difficult than the reverse.

1.1 MODELLING

Understanding concepts and their interrelations is greatly facilitated via the use of modelling tools (Orfanos & Dimitracopoulou, 2003), taking into account that the modelling process forces students to change their vague, imprecise ideas into explicit causal relationships (Niedderer et al, 1991).

Moreover in traditional teaching, many concepts are introduced in a sequence. The above doesn't supply students with all the necessary information in order to understand the concepts. On the contrary, in a modelling environment, the concepts are related among them, with the help of tools provided by the software. The function can be controlled by the outcome of the relationship of certain selected concepts. The result is subject to criticism as compared to the forecast, and if it is not desired, the learner can either make changes in the model, or create another model in order to test it again. This process can be repeated until the model will give the desired result. It is possible to check only one relationship between the properties, blocking the function of other links, in case there is more than one relationship.

The ability to identify and represent the same concept through different representations is considered as a prerequisite for the understanding of the particular concept (Duval, 2002; Even, 1998). Some researchers interpret students' errors as either a product of a deficient handling of representations or a lack of coordination between representations (Greeno & Hall, 1997). The standard representational forms of some mathematical concepts, such as the concept of function, are not enough for students to construct the whole meaning and grasp the whole range of their applications.

For this purpose, we have used ModellingSpace, an environment which was especially designed to allow students from eleven to seventeen years old to express their ideas and gradually develop them. The ModellingSpace enables students to build their own models and offers the choice to observe directly simulation of real objects and/or all the other alternative forms of representations (tables of values, graphic representations and bar-charts). The students formulate hypotheses, (in order to answer the questions), creating models, comparing their hypotheses with the representations of their models and modifying other models when their hypotheses do not agree with the representations.

Even (1998) focused on the intertwining between the flexibility in moving from one representation to another and other aspects of knowledge and understanding. This study indicated that subjects had difficulties when they needed to flexibly link different representations of functions, students can plot and read points but cannot think of a function in a global way. Who can easily and freely use a global analysis of changes in the graphical representation have a better and more powerful understanding of the relationships between graphical and symbolic representations than people who prefer to check some local and specific characteristics, (Monoyiou & Gagatsis, 2008).

2 RESEARCH

This current research project was conducted as a part of doctoral dissertation. The main focus of the research concerns some in-depth investigation of school student groups involving a series of activities. Eleven third grade high school pupils who participated were divided into 5 teams, in most of the activities on a voluntary basis. Our effort was to accurately record the reactions of the conversations and actions of students in order to analyse the contribution of modelling activities in learning physics and cognitive processes involved. This paper focuses on overcoming any obfuscation of the mathematical equations.

We present the parameters that obscure and make it difficult for students to get along certain functions, especially in Physics. Activities were based on the Scaffolding method. Consequently, the aims evolved in response to the investigation progress. Initially we wanted them to create models with semi-qualitative relations, in order to discover physics magnitudes and connect them conceptually with magnitudes they already had been taught. While in the final activities we wanted students to work with quantitative relations and to manage quantitative reasoning, to draw up the equations for the position, to understand every element in the equations of motion as the degree, the sign that connects each term, the fixed factor in each term of the equation.

The surveys were designed with specific modelling activities with reference to kinematics for high school students. These activities by utilising the possibilities of realistic simulation modelling tool and by enhancing students to interact with each other, contribute to conceptual change and to the accomplishment of deeper understanding of concepts, (Orfanos & Dimitrakopoulou, 2003).

Students participating in modelling activities create models, predict, and evaluate their models and, most importantly, compare their predictions with the models' representations. They have the opportunities to observe the results of regulations that they make in the magnitudes relations which connect magnitudes in the graphic representation, on the table of values and in the simulation. The students observe the results tangibly and they can easily realise by themselves the alternative ideas that they may formulate for the relation of proportion or for the constant 's sign. With the aid of modelling activities the students deal with complex situations without having to do routine work, such as numerical calculations, or to manufacture many graphs, to really think in terms of scientific variables, to understand the transformations of the situation under study into relational terms, (Smyrnaiou & Weil-Barais, 2003), to choose the most appropriate representation/s on screen, and thus to solve more complex problems.

The parameters on which they had additional problems in promoting a further understanding of the functions in physics, concern mainly a) the factors of formulas, b) the additional meaning of the signs in equation in physics and c) in the functions' content.

3 DISCUSION

3.1 DIFFICULTIES CONCERNING THE NATURE OF FUNCTIONS

Function is a mental construction that was integrated rather recently in mathematics. The notion of function is so abstract that presents many difficulties in its didactical metaphor. Different epistemological approaches that led to the meaning of function through its long historical evolution are disrupting into the teaching guides and textbooks of mathematics in a confusing way. The complexity of this didactical metaphor has been a main concern of mathematics educators and an active question in the research of mathematics education. Researchers usually investigate the epistemological obstacles, on the basis of the historical study of the concept of function, and propose teaching methods, which aim at overcoming these obstacles. In practice, different approaches that are applied in mathematics instruction concerning the concept of function result in exposing to the students the pieces of a puzzle consisting of a vague set of extracted information, that possibly merge at university level in mathematics. Sierpinska (1992) gives a viable example of such an approach supporting that formulae, graphs, diagrams, word descriptions of relationships and verbal expressions, compose an uncertain schema of thoughts (Evangelidou, Spyrou, & Gagatsis, 2004)

Students' difficulties had to do chiefly with the function itself. We found that students seemed to have difficulties in understanding the dependent and independent variables, as well as the proportional and the constant function.

Van Dooren, De Bock, Janssens and Verschaffel (2005b) referred that, proportionality seems to be a belief deeply rooted in the intuitive knowledge of the students used spontaneously unconsciously. Because of its wide applicability for understanding mathematical, scientific and everyday life problems, linearity (or proportionality) is a key concept throughout primary and secondary mathematics education. Inherent to the attention it receives, however, is the risk to develop an over reliance on the concept: “Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as if it were linear” (Freudenthal, 1983, p. 267). The tendency to overgeneralise the linear model is repeatedly mentioned in the mathematics education literature, and in recent years it has also been in the focus of systematic empirical research. For example, the phenomenon has been studied in elementary arithmetic (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005), algebra and calculus (e.g., Esteley, Villareal, & Alagia, 2004).

The relation of proportion is one of the first relations the students use. The students comprehended the relation of the position proportional to the time from the stroboscopic photographs, from the verbal description of rectilinear uniform motion (every time they could recognise the type of motion) as well as from the pattern of the graphic representation and they filled in the magnitude values of the table when the constant of proportion had the value one.

One team commenting on the change in position relatively to time using a table values where the displacement values were always the same, suggested that it "varies with time” while the expected answer was: The change of position is constant or always 10.

When asked on “what is concluded from the graph when we minimise the constant ratio”, another group responded: "the time is not proportional to the position if we minimize the constant ratio”.

In pilot studies, the students produced a model of the position by way of time. When they were asked to create a model for the same movement with velocity and time properties, a group created model again with the relation of proportion, instead of the constant.

The students faced difficulties in recognising the relation of proportion when the constant of proportion was different from one. They didn’t answer correctly about the magnitude values of the table and they didn't correspond the right graph to the relation of proportion. The students' difficulties that we noticed for the relation of proportion were:

The students answered wrong for the magnitude values of the table.

The students confused the constant function with that of proportion.

The students predicted incorrectly on the type of function which would apply if the value of the constant ratio changed.

The students understood the relationship of proportion, which they felt rather more accessible to them. They understood the role of the constant ratio, they saw that the change of the constant ratio made the rider run faster. At first they could see without associating the constant ratio of the equation position's time with the sense of velocity. With the contribution of representations they could associate higher velocity with the concept of faster speed. In the completion of the activity, the students managed to associate the constant-ratio when the position was proportional to time to associate it with velocity.

The student groups used different strategies in the activity that aimed at identifying the velocity with the constant of the proportion, in the relation of position to time. A team utilised the representation of table of values in order to discover the role of velocity in the model and connected intellectually the rate of change of position with the velocity. Another team observed the simulation, and managed to make a motorcyclist run faster, by altering the constant of proportion. This later comprehended deeply the constant of proportion, connecting intellectually the constant of proportion with the velocity without assimilating them completely.

In addition students created easily a model with a semi-quantitative relationship (the position proportional to time) looking through the graph and justifying the analogy as follows: “when time increases the position increases too and the magnitudes are proportional"; they also supplemented the table values' correctly. Most groups understood that the constant ratio is the slope of the graph.

Moreover in subsequent quantitative activities we noticed that they used a more accurate wording: "The displacement is constant over time at equal intervals”. This indicates that they could express the change of the magnitude of position in relation with the magnitude of time. This finding also indicates that they understood the deeper meaning of the function, i.e. which magnitude is independent and which dependent.

Through the lens of modelling activities, the students fathomed more in the relation between magnitudes and in their representations. The students comprehended more deeply the constant function and managed to distinguish the second degree function from the relation of proportion. They also realised that there are other relations beyond those, which are being taught in the school class. The students understood more deeply the concepts and their representations, which constitute the organisation of the concept towards the scientific theory.

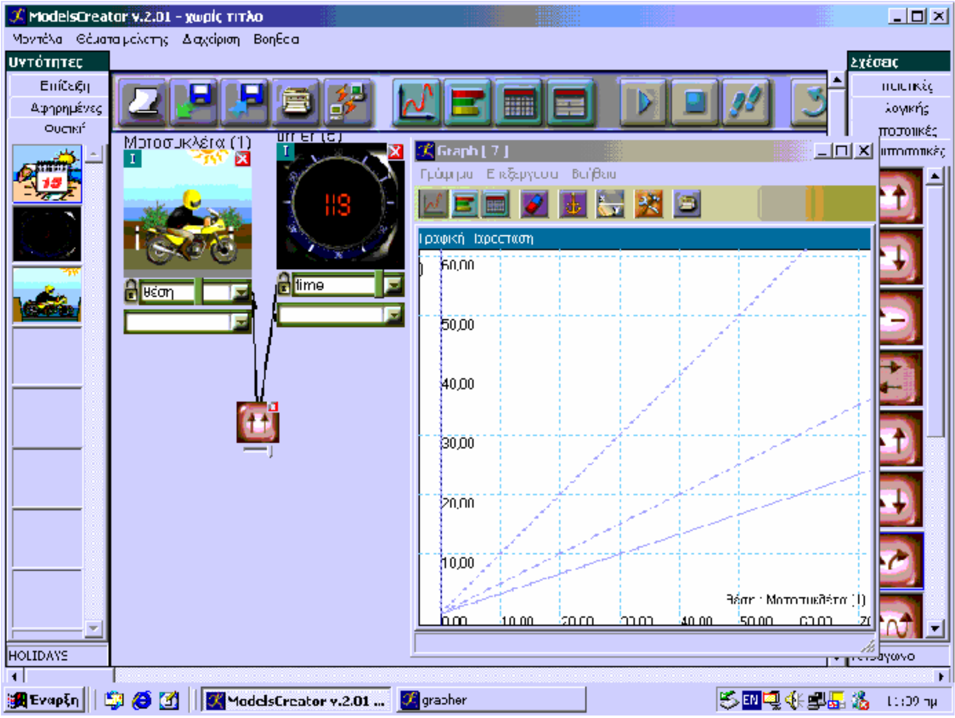
Over the process skill "comprehension of the constant of proportion role", the students following the modelling cycle prediction-observation-revision, discovered the role of the constant of proportion in the relation of position to time, as on the graph, on the table and in the simulation of model.

3.1.1 Independent-Dependent Variables

White and Mitchelmore (1996) showed that students have a primitive understanding in the concept of a variable. The study involved four questions and each question had four versions. Version A required the students to do more translation from English to mathematical symbolism while version D required them to do very little translation. These questions were used in their research performed on first year university students, all of whom had studied calculus in secondary school. They found that students treated variables as symbols to be manipulated rather than as quantities to be related. In the problems that were given, the students were unable to distinguish between a general relation and a statement of a specific variable. This underdeveloped concept of a variable made it difficult to identify and symbolize an appropriate variable by translating one or more quantities and therefore define a usable function. Eisenberg, T. (2006) referred that the student do not understand the relationship the graph describes between the independent and dependent variables.

The students considered about the dependent / independent variables in the relationships. The difficulty with the independent and dependent variable was noticeable for students who did not participate in all activities. The expression of students "if we double the time" shows that they considered the independent variable as a magnitude that the user could change not as an independent one.

In order to be able to interpret scientifically phenomena of everyday life, students initially needed to distinguish the cause from its effect as well as the independent from the dependent variables. In ModellingSpace, the user has to distinguish the independent variable that is discerned by the type of lines. In model of Figure 1, the 'time' is the independent variable. The students' worksheets have been designed in such a way so as that students think about the type of the variables and the relation connecting them in order to proceed-for a start-to a rough scientific organization of the concepts at issue. Understanding the relationship of a variable relative to time is a prerequisite for the understanding of the variable 's change relative to time and for enabling to proceed in quantitative reasoning. The question: "How much does the position change every second?" seemed to be



easier than the question: "What can you observe in the change of its position?"

Figure 1. Screen-shot from a student 's model. It was executed 3 times with different values in constant ratio.

Analysing the corresponding research findings, we noticed that the students couldn't distinguish the independent from the dependent variables, especially during the initial activities. By quoting some of the students' dialogues in chronological order, we can notice the positive contribution made by the modelling activities to the deeper understanding of the dependence of the first magnitude's values on those of the second one. In the first activity the student "Giorgos" made the following assumption. "Let say that time is proportional to position. Is this the same as saying that position is proportional to time?" A week later "Giorgos" worked with another student "Tasos" who was not present in the previous activity. They were both asked to fill in some missing values in a table constructed to relate position to time and then to explain how they reasoned in order to accomplish their task. "Tassos" justified their choice saying that they opted for the specific values "because time is proportional to position". After this answer, the following dialogue followed:

S232 Giorgos: Because position is proportional to time.

S233 Tassos:	Oh! Yes. Because time is proportional to position.
S234 Giorgos:	Because position is proportional to time.
S235Tassos:	Why?
S236 Giorgos:	Because position is proportional to time.
S237 Tassos:	...
S238 Giorgos:	What are you saying; No one can affect time.

It seems that “Giorgos” had understood the concept of the magnitude-dependence qualitatively and in depth. He didn't recollect his answer and he used sound arguments with a view of convincing his colleague. Another characteristic example is what another student “Anne” said: “Position is proportional to time… or time is proportional to position? Oops, what am I and it wasn't long before saying?”, we noticed that her team linked the variables correctly.

The students initially don't distinguish the independent from the dependent concept. Researchers such as Screen (1986), Adey et al (1995), Foss (2000) consider it as sub-skill lying in the hierarchy lower than the skill of controlling variables. In the present research, the percentage of students that cannot distinct the dependent–independent variables decreased with the progress of activities. The frequency of appearance of this alternative idea was also decreased. In latest activities this difficulty did not appear.

3.1.2 Constant function

Tall and Bakar (1992) comparing student performance on the expression y = 4 and the graph of y = constant, they found that there is evidence of conflict in a significant number of scripts, as students change their mind when realizing that the algebraic expression clearly does not involve x, but the graph seems more likely to be a function. Sand (1996) referred that students have trouble grasping this concept [of many-to-one] in the earlier stages of function work because school textbooks do not put much emphasis on constant functions.

Students of that age, or even elder as we saw in the pilot-studies, present some difficulties in understanding the constant function. In semi-quantitative relations the facility in using the relation of the constant made us think they understood the constant function. But when they had to compose a quantitative relationship it was revealed that their concern was not only the drafting of the relation. Their dialogues showed that they hadn't understood exactly what constant function or the value of the variable is constant, while the value of time changes continuously meant. Students focused only on one magnitude, i.e. it seemed that the difficulty lies on the understanding of the relation of the concepts and even more when one remains in the same value.

Furthermore in pilot studies we saw that initially they tried to identify speed=constant. Even from pilot studies, the students' first response was that when the magnitude is fixed, its value is zero.

In their first activities the students studied what happened in one magnitude and not what happened to a magnitude relatively to the other. For example they had understood that velocity determines the change of position, but they couldn't refer to it as rate of change, and they couldn't indicate time in some way for example: quickly or slowly. Several times they used the concept of proportional instead of fixed function.

With the following activities and the teacher's contribution the students discovered how to draw the constant function with the help a fixed actual price, such as 20km/h for speed. As the activities progress we observe that they are more comprehensive about the function of the magnitudes.

3.1.3 Regarding physics

In physics there are extra parameters of difficulty in understanding functions. The factors in the formulas in physics are concepts which relate with each other or with other concepts, so a formula in physics is a complex mental construction, which is difficult for students to comprehend. Moreover in many cases the factors in a function may be additional functions and this adds for students even more difficulty in understanding it, e.g. in the function of position $x = \upsilon t$ the velocity υ is another function of time t.

Rozier and Viennot (1991) also see students treating variables in a primitive manner. Their study showed how students reduce the number of variables, or take all the variables into account, but in a simplified way, when dealing with thermodynamic problems. Because of a "preferential association" between two variables, we see students relating only these two variables and ignoring the others. Rozier and Viennot also see students reducing the number of variables by combining two variables and treating them as one. Linear causal reasoning is another way students are able to deal with only two variables at a time even when the "changing physical quantities are all supposed to change simultaneously". This linear reasoning involving successive steps allows the student to relate two variables while keeping the others constant during each step. Thinking about more than two variables at a time seems to be a very difficult task for students and this difficulty surfaces in various topics such as thermodynamics.

We studied the way students connected and integrated the symbolic (quantitative) description of a magnitude with the formulas of motion and vice versa. We also studied whether they understood the meaning and content of mathematical relationships and equations in physics. Glimpsing into two functions of the same degree that have different contents when the magnitude of the dependent variable is changed is another aspect that deserves closer and more attentive scrutiny, e.g. the velocity u = constant refers to a different type of motion than the acceleration a = constant. This is to say that two functions which are equivalent in mathematics have different meanings in physics when they are in different physical magnitudes. In physics these functions will be simulated differently. The equation $\psi=12-3\chi$ is a decline mathematical function, but when used as a function of velocity $\upsilon=12-3t$ describes two different motions. The first is a decelerated motion (by the time 4) and the other is an accelerated motion (after the time 4).

Specifically, we investigated students' understandings of each element of the equation, such as the function's degree, the sign connecting each term, and the values of fixed terms. We groped whether and how students connected the magnitudes on the type of motion and the contribution in understanding of their forecasts for simulation or other representation of some equation by comparing those of the software, the final aim being once they understood the above mentioned concepts and interrelationships, to be able to write the appropriate function, given the description of motion.

Eisenberg (2006) refers that although there are many facets to mathematical anxiety, notational complexities are often obstacles in preventing understanding of function concepts. Again we meet the problem that it is not the mathematics, but the representation of the mathematics. Notational difficulties sneak in everywhere in elementary mathematics.

On several occasions the mathematical notation has a different meaning in physics than in mathematics, leading students to alternative ideas. The role of the sign has a different meaning when referring to different physical quantities: for example if car 's (A) velocity is 20m/s and car 's (B) velocity is -30m/s, car A runs at lower velocity than B. On the contrary in its mathematically equivalent is applied as 20m/s>-30m/s.

The sign in the velocity indicates the direction of motion, the sign in the acceleration compared with the sign of velocity indicates the kind of motion (accelerating or decelerating) and the sign in the position indicates the position related to the starting point.

Students were asked to create a model and using the equation editor to write the relationship x=10 +4t for a motorcyclist who is moving linearly. They were asked to answer questions and then checked their answers by using the model and its representations: simulation, indicating property values, graphs and table values. The questions were: "What is the kind of the car's motion, what is the initial position of the car, what is the velocity of the car and what is the acceleration of the car?"

The students could not tell the type of motion using the equation x=10+4t, whereas two of the three groups and initially the third group responded linear accelerated motion. Only after they ran the model with the help of the graph and table values recognized that the type of the motion is rectilinear uniform motion.

From the position's function the students at first recognized just the magnitudes of the position and time; they did not recognize the velocity, initial position and acceleration. Subsequently, they discovered these magnitudes in the equation of the position with the help of the implementation and testing of the model and its representations.

They realized that the constant term in equation (here 10), indicates the starting position. One student said she saw the graphics, but she couldn't convince her classmate, who only agreed after she noticed the values on the table.

They found the velocity by dividing the values of the position by the corresponding time (x/t) and not those of position's change through the time intervals ($\Delta x/\Delta t$). Although they have written $\upsilon=\Delta x/\Delta t$, they had extinguished it but they wrote it back. They were also wrong in calculating the velocity because they used formulas of linear uniform motion without proper starting position with the motion as a starting position. They didn't use the general formulas although they had already said and had written them down.

They deleted and rewrote the relation in their model because they didn't like the graph, they had different axes originally, namely, the time on the vertical and position on the horizontal axis. In an effort to discover what they hadn't done well, they wrote the same equation, with a different order in factors the fixed term at the end: $x = 4t + 10$.

They wrote the formula of acceleration as $a = \upsilon/t$, (not the ratio of the magnitudes' changes). The formula $a = \upsilon/t$ is used only in rectilinear uniformly accelerated motion without initial speed. In their dialogue they wondered: "The acceleration is equal to $\Delta\upsilon/\Delta t$?" Although they knew the formula they didn't implement it properly. Characteristically, they replied that they had written the formula 10 times. It was repeated the same that had happened in calculation of the velocity, i.e. they didn't use the general formula.

The representations of the model and particularly, in this activity, the simulation (move to the left), gave the students an opportunity to reflect on the model, to explore the model and specifically the function, to discuss this issue more closely with each other and with the researcher, modify the function in order to understand the role of factors. They observed both the model and its simulation simultaneously. This was rare, when the simulation was presumed from the students. The simulation caught their interest in order to work on the issue without being bored or disappointed. By making changes in function, they observed different representations that puzzled them and discussed about them with each other or the researcher.

The cognitive process employed by the students is as follows: Firstly, they understood the role of each factor in the equation. The identification of each factor in the equation with the physical corresponding magnitude is of a higher degree cognition and needs more effort to be achieved.

The initial models created by students did not interpret the phenomenon as a whole; every relationship refers to something from the model. Thus the functions among the properties were independent from each other; each function connected a magnitude with time. The activities' aim was to link all magnitudes with one function, building and deepening the understanding of the concepts in the sense that expresses the function between them. Thus, finally, they would express it in a quantitative manner.

The modelling helped our students to link three concepts; position, velocity and time in one function with the Scaffolding process as follows: Initially they maintain the velocity but didn't consider it necessary to include it in the equation. The argument, which is reported as an excuse was that velocity was constant. With the help of the tests on their model, as indicated by the researcher, the students, by using their model as the scaffold, found out that the velocity, if not written into the equation, didn't play a crucial role in the model and its representations. The students went through the tests to import velocity into the equation.

Depending on the activities' question the students handled the desktop. When asked to modify the model so that the motorcyclist to move opposite, they changed the value of velocity to be negative and they observed in the simulation the motorcyclist to change direction and move to the left. Now they had both the model and the window of the graph to be visible. They focuses their attention to the direction of the motion. In previous questions they had the window of the graph over the model.

The syntax of the function requires higher-order reasoning, than that of using the formula for calculating the value of an unknown magnitude. Students know the formulas but couldn't structure the quantitative relationship; they worked in the direction to calculate the value of a magnitude. Some difficulty was noticed in finding the type of arithmetic operation to put motorcyclist 's original position in the equation of motion. They found that it was needed to complete the equation with the initial position but they used the operation of multiplication. At this point the graph representation was utilized.

There was given an opportunity to students to think and express their views on key issues that are often considered self-evident and obvious. It is likely that even teachers found them self-evident. Even the students themselves discovered the points that didn't understand. The students recognized easily if they choose a non appropriate function, but it was not very easy for them to realize which concepts were wrongly connected.

They understood that the magnitudes were inerrelated and that by using the function of a magnitude to time someone could find another magnitude; even when this magnitude isn't referred in the function.

The students initially tried to comprehend the magnitudes and then the connection or identification between the magnitudes and the factors of the function such as the constant ratio with the velocity in the uniform rectilinear motion.

During the initial activities the students studied what happens in one quantity and not what happens to a quantity relatively to the other. E.g. they understood that velocity determines the change of position, but they didn't refer to it as rate of displacement, they didn’t even indicate the time at anyhow (using adjectives such as quick). In the progress of the activities they comprehended the deeper meaning of the function. They comprehended the function between two magnitudes, while, initially, they expressed the change of one magnitude, before they expressed the relation in the magnitude over time.

They continued to recall for a long time, what they had learned from previous activities. They understood that there are motions beyond those which they had been taught.

4 CONCLUSIONS

In all activities we invited the students' reasoning to pass from qualitative to the semi-quantitative and then to the quantitative one. Understanding the qualitative relationship between the variables is a prerequisite for understanding the change of the magnitude in terms of time. Students are facilitated when the quantitative models follow the semi-quantitative ones, they understand the relationship of change, initially by using semi-quantitative reasoning, before they can use the quantitative reasoning.

Concepts not yet understood by the students, as well as inappropriate mental representations are especially difficult to trace and students cannot therefore manifest them, unless they are invited to deal with situations that require high order thinking. Situations of this kind appear more often in modelling activities than in traditional education.

Activities help to highlight issues that are not understood by students, such as constant and proportional function, the independent and dependent variables in order to try to understand them. Students initially understand the part of each factor in the functions and then they attempt to associate it with the concepts of physics. Students utilized the graphs to understand them in depth, to find magnitudes, to supplement or modify their model.

Many researchers refer on the importance of multiple representations in teaching and learning of mathematics: Elia & Gagatsis (2006) say that, nowadays the centrality of different types of external representations in teaching and learning mathematics seems to become widely acknowledged by the mathematics education community. Duval (2002) using multiple representations for the same mathematical situation is at the core of mathematical understanding. Ainsworth, Bibby, and Wood (1997) suggest the use of multiple representations can help students develop different ideas and processes, constrain meanings and promote deeper understanding. Interacting with multiple representations requires the understanding of the relationship between them which is a complex process. Our research findings agree with the above. The diversity of representations and realistic simulation contributes to the understanding of quantitative reasoning in contrast to traditional teaching where representations obstruct understanding.

The whole learning environment (technology based learning environment, modelling activities, students' worksheets) seems to support students progressively thereby supporting researchers as stated in: "The modelling activities help the students revise their alternative ideas that sometimes are deeply entrenched and prevent them from grasping the concepts of physical sciences" (Vosniadou, 1994). "Modelling activities contribute to the creation in the students' minds of a world of concepts that accords with Physics' world" (Hestenes, 1992). "The [modelling] activities described below constitute a first attempt towards creating learning situations in a computational environment, which provide opportunities for promoting pupils' mathematical understanding through the development of their modelling skills (Sakonidis, 2003)".

According to Eisenberg (2006) functions and their associated notions are not conceived visually, and that this non-visual approach hinders one's development of having a sense for functions. Students seem to think of function concepts in only a symbolic representational mode. It is a serious impediment to students' learning. We think that learning environments such that we used in our research, which can make visible the representations of functions and the changes in function parameters may contribute to a deeper understanding of functions by the students.

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