The transversality condition associated with the unspecified final time variation δt_f is

$$\frac{\partial P'}{\partial t_f} + H_f + \sum_{i=1}^{7} \frac{\partial^2 P'}{\partial x_{i_f}\partial t_f}(\delta x_{i_f} + \bar{g}_{i_f}\delta t_f) + \frac{\partial^2 P'}{\partial t_f^2}\delta t_f + \sum_{i=1}^{7}\frac{\partial H}{\partial x_{i_f}}(\delta x_{i_f} + \bar{g}_{i_f}\delta t_f) + \sum_{i=1}^{7}\frac{\partial H_f}{\partial \lambda_{i_f}}(\delta\lambda_{i_f} + \dot{\lambda}_i\delta t_f) + \frac{\partial H_f}{\partial \alpha_f}(\delta\alpha_f + \dot{\alpha}\delta t_f) + \frac{\partial H_f}{\partial \theta_f}(\delta\theta_f + \dot{\theta}\delta t_f) + \frac{\partial H_f}{\partial t_f}\delta t_f + l_f\delta t_f = 0 \quad (27)$$

Except for the last term, which corresponds to the constraint Eq. (13), Eq. (27) is a linearization of the transversality condition

$$\partial P'/\partial t_f + H_f = 0 \quad (28)$$

The first-order necessary condition corresponding to a switch between thrust levels at time t_j is

$$(H^- - H^+)_j \equiv \lim_{\epsilon\to 0}[H(t_j - \epsilon) - H(t_j + \epsilon)] = 0 \qquad j = 1,2 \quad (29)$$

Equation (29) is equivalent to the vanishing of the "switching function" $\partial H/\partial T$. The corresponding necessary condition for the accessory problem is a linearization of Eq. (29) plus a "gradient term," corrsponding to the step-size constraint [Eq. (11) or (12)]:

$$(H^- - H^+)_j + \delta(H^- - H^+)_j + l_j\delta t_j = 0 \quad (30)$$

In the refinement process described in Ref. 1, terminal conditions for the accessory problem consist of linearized versions of the terminal conditions [Eqs. (19) to (23)] and the linearized version of the transversality condition

$$(\lambda_4 y + \lambda_1 v - \lambda_5 x - \lambda_2 u)_{t_f} = 0 \quad (31)$$

corresponding to freedom of position along the terminal circular orbit.

At thrust switching times other than rocket staging times, the state variables are continuous, and considerations arising in the classical analysis of "corners" apply. These considerations lead to continuity of the multiplier variables. The total variations of state and adjoint variables on each side of a switching time t_j must then be equal:

$$(\Delta x_i^- - \Delta x_i^+) = (\delta x_i^- + g_i^-\delta t_j) - (\delta x_i^+ + g_i^+\delta t_j) = 0 \quad (32)$$

$$(\Delta\lambda_i^- - \Delta\lambda_i^+) = (\delta\lambda_i^- + \dot{\lambda}_i^-\delta t_j) - (\delta\lambda_i^+ + \dot{\lambda}_i^+\delta t_j) = 0 \quad (33)$$

These equations imply jump discontinuities in the velocity and mass variations $\delta\mu$, δv, δw, and δm at the switching times and imply a discontinuity in the adjoint-variable variation $\delta\lambda_m$ determined by these equations.

Analytical Treatment of Coasting Arcs

Solutions of the two-body central force problem, the corresponding adjoint, and the transition matrices for state and adjoint variations are computed in closed form in Cartesian coordinates, in terms of universal variables (i.e., variables suitable for elliptic, parabolic, and hyperbolic orbits). The state vector at time t may be expressed in terms of the state at some epoch time t_0 by means of the classical f and g series defined in Eqs. (37–40), and the first and second partial derivatives of the state with respect to the initial state components may be found. It is necessary, as a preliminary step, to obtain the generalized eccentric anomaly β, which is the independent variable of the series expansion.

The universal variables arise from a transformation of coordinates which removes the second-order singularity that occurs in the two-body equations at the center of attraction. The generalized eccentric anomaly β is defined implicitly by

$$d/d\beta = (r/\mu)(d/dt) \quad (34)$$

where r is the magnitude of the position vector, μ is the gravitational constant, and β is assumed zero at the epoch time t_0. In the case of elliptic orbits, β is $a^{1/2}(E - E_0)$, in which a is the semimajor axis, and E is the eccentric anomaly.

The universal variables are transcendental functions defined by the series

$$G_n = \beta^n\left[\frac{1}{n!} - \frac{\alpha^2}{(n+2)!} + \frac{\alpha^4}{(n+4)!} - \ldots\right] \quad (35)$$

where $\alpha^2 = \beta^2/a = -\gamma\beta^2$, $\gamma = -1/a = V_0^2/\mu - 2/r_0$, and V is the magnitude of the velocity vector. Thus, these G_n are expansions in β^2 which must be truncated for computation.

The anomaly β is the solution of the generalized Kepler equation

$$\hat{M} = G_3 + r_0 G_1 + (d_0/\mu^{1/2})G_2 \quad (36)$$

where $\hat{M} = \mu^{1/2}(t - t_0)$ and $d_0 = R\cdot V$. The algorithm used computationally for the solution of this equation is Newton-Raphson iteration.

The f and g representation coefficients are

$$f = 1 - G_2/r_0 \quad (37)$$

$$g = (t - t_0) - G_3/\mu^{1/2} = (\hat{M} - G_3)/\mu^{1/2} \quad (38)$$

$$\dot{f} = -\mu^{1/2}G_1/rr_0 \quad (39)$$

$$\dot{g} = 1 - G_2/r \quad (40)$$

The state at time t is expressed in terms of $x_i(t_0)$, f, g, $\dot{f}$, and $\dot{g}$ by

$$x_i(t) = fx_i(t_0) + gx_{i+3}(t_0) \quad (41)$$

$$x_{i+3}(t) = \dot{f}x_i(t_0) + \dot{g}x_{i+3}(t_0) \quad (42)$$

The development used originally[4] was restricted to elliptic orbits, but has been superseded by the universal variable development sketched here and reported in detail in Refs. 2 and 5. Closed-form expressions for the λ_i, δx_i, and $\delta\lambda_i$, bridging the coast, are given in Refs. 2 and 5.

Choice of the Weighting Functions k_α and k_θ

A necessary condition for the existence of a minimum of the function given by Eq. (8), subject to the constraints previously discussed, is that the matrix

$$Q = \begin{bmatrix} l_\alpha k_\alpha + \dfrac{\partial^2 H}{\partial \alpha^2} & \dfrac{\partial^2 H}{\partial\alpha\partial\theta} \\ \dfrac{\partial^2 H}{\partial\alpha\partial\theta} & l_\theta k_\theta + \dfrac{\partial^2 H}{\partial\theta^2} \end{bmatrix} \quad (43)$$

be positive semidefinite, which corresponds to the existence of a minimum of the function $h(\delta\alpha,\delta\theta)$ given by Eq. (18). Although this is necessary, it is by no means sufficient, since the existence of a minimum for the quadratic-linear variational problem only could be guaranteed by the satisfaction of a strengthened Jacobi-like condition appropriate to a variational problem including control parameters. In general, such a condition would require, indirectly, that the step-size control parameters $l_\alpha k_\alpha > 0$, $l_\theta k_\theta > 0$, $l_1 > 0$, $l_2 > 0$, and $l_f > 0$ be "sufficiently large." For very large values of these quantities, it can be shown that the penalty-function version of the successive approximation process becomes a gradient process, and hence that convergence is assured, in the sense that a direction of decreasing P' is found. Strictly speaking, the suitability of choice of the constraint multipliers at each step of the successive approximation process could be verified only by performing, compu-