In the case when $\bar{\mu}(E(K)) \ge 3/4$ we have $\bar{\mu}(K - E(K)) \le 1/4$. By the case $n = 1$ above without loss of generality $\bar{\mu}_x(pr_x(E(K))) \ge \bar{\mu}(E(K))/2$. Hence $|\mu_x| \ge \frac{1}{2}\cdot\frac{3}{4} - \frac{1}{4} = \frac{1}{8}$, thus (*) holds for $\varepsilon = \frac{1}{8}$. □

Smoothly basic subsets of the plane.

Let K be a subset of the plane $\mathbf{R}^2$. A function $f : K \to \mathbf{R}$ is called *differentiable* if for each point $z_0 \in K$ there exist a vector $a \in \mathbf{R}^2$ and infinitesimal function $\alpha : \mathbf{R}^2 \to \mathbf{R}$ such that for each point $z \in K$

$$f(z) = f(z_0) + a \cdot (z - z_0) + \alpha(z - z_0)|z, z_0|.$$

Here the dot denotes scalar product of vectors $a =: (f_x, f_y)$ and $z - z_0 =: (x, y)$, i.e. $a\cdot(z - z_0) = xf_x + yf_y$. A function $\alpha : \mathbf{R}^2 \to \mathbf{R}$ is *infinitesimal*, if for each number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each point $(x, y) \in \mathbf{R}^2$

$$\text{if} \quad \sqrt{x^2 + y^2} < \delta, \quad \text{then} \quad |\alpha(x, y)| < \varepsilon.$$

Let V be the graph of the function $y = |x|$, where $x \in [-1; 1]$. A function $f : V \to \mathbf{R}$ is differentiable if and only if $f(x, |x|)$ is differentiable on the segments $[-1; 0]$ and $[0; 1]$.

A subset $K \subset \mathbf{R}^2$ of the plane is called *differentiably basic* if for each differentiable function $f : K \to \mathbf{R}$ there exist differentiable functions $g : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

13. (a) (b) (c) Solve the analogues of problem 6 for differentiably basic sets.

14. (a) The graph V is differentiably basic.

(b) $W := (V - (2, 0)) \cup (V + (2, 0))$ is not differentiably basic.

(c) The broken line whose consecutive vertices are $(-2, 0)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 0)$ is not differentiably basic. (Note that it is continuously basic).

(d) The completed array $\{([\frac{n+1}{2}]^{-1/2}, [\frac{n}{2}]^{-1/2})\}_{n=2}^{\infty} \cup \{(0, 0)\}$ is not differentiably basic. (Note that it is also not continuously basic.)

(e) The completed array $\{(2^{-[\frac{n+1}{2}]}, 2^{-[\frac{n}{2}]})\}_{n=1}^{\infty} \cup \{(0, 0)\}$ is differentiably basic. (Note that it is not continuously basic.)

(f) (I. Shnurnikov) The cross $K = [(-1, -2), (1, 2)] \cup [(-1, 1), (1, -1)]$ is not differentiably basic. (This assertion and Conjecture 15a imply that the property of being differentably basic is not hereditary.)

(g) If a graph is basically embeddable in the plane, then it is differentiably basically embeddable in the plane. (This is non-trivial because the plane contains graphs which are basic but not differentaibly basic and vice versa.) [RZ06]

15. Conjectures.** (a) (I. Shnurnikov) A completed array $\{a_n\}_{n=1}^{\infty} \cup \{(0, 0)\}$ is differentiably basic if and only if the sequence $\frac{\sum_{n=k}^{\infty} |a_n|}{|a_k|}$ is bounded.

(b) The subset $\{(t^2, \frac{t^2}{(1+t)^2})\}_{t\in[-\frac{1}{2};\frac{1}{2}]}$ of the plane is not differentiably basic.

Hint. One can try to prove this analogously to 14f. Cf. [Vo81, Vo82].

(c) A piecewise-linear graph in $\mathbf{R}^2$ is differentiably basic if and only if it does not contain arbitrary long arrays and for each two singular points a and b we have $x(a) \ne x(b)$ and $y(a) \ne y(b)$. A point $a \in K$ is *singular* if the intersection of K with each disk centered at a is not a rectilinear arc.

It would be interesting to find a criterion of being differentiably basic for closed bounded subsets of the plane. Apparently a simple-to-state criterion (analogous to the Sternfeld criterion) does not exist. Another interesting question: is there a continuous map $[0; 1] \to \mathbf{R}^2$ whose image is differentiably basic but not basic?