In the case when  $\bar{\mu}(E(K)) \geq 3/4$  we have  $\bar{\mu}(K - E(K)) \leq 1/4$ . By the case n = 1 above without loss of generality  $\bar{\mu}_x(pr_x(E(K))) \geq \bar{\mu}(E(K))/2$ . Hence  $|\mu_x| \geq \frac{1}{2} \cdot \frac{3}{4} - \frac{1}{4} = \frac{1}{8}$ , thus (\*) holds for  $\varepsilon = \frac{1}{8}$ .

## Smoothly basic subsets of the plane.

Let K be a subset of the plane  $\mathbf{R}^2$ . A function  $f: K \to \mathbf{R}$  is called *differentiable* if for each point  $z_0 \in K$  there exist a vector  $a \in \mathbf{R}^2$  and infinitesimal function  $\alpha: \mathbf{R}^2 \to \mathbf{R}$  such that for each point  $z \in K$ 

$$f(z) = f(z_0) + a \cdot (z - z_0) + \alpha(z - z_0)|z, z_0|.$$

Here the dot denotes scalar product of vectors  $a =: (f_x, f_y)$  and  $z - z_0 =: (x, y)$ , i.e.  $a \cdot (z - z_0) = x f_x + y f_y$ . A function  $\alpha : \mathbf{R}^2 \to \mathbf{R}$  is infinitesimal, if for each number  $\varepsilon > 0$  there exists a number  $\delta > 0$  such that for each point  $(x, y) \in \mathbf{R}^2$ 

if 
$$\sqrt{x^2 + y^2} < \delta$$
, then  $|\alpha(x, y)| < \varepsilon$ .

Let V be the graph of the function y = |x|, where  $x \in [-1; 1]$ . A function  $f : V \to \mathbf{R}$  is differentiable if and only if f(x, |x|) is differentiable on the segments [-1; 0] and [0; 1].

A subset  $K \subset \mathbf{R}^2$  of the plane is called *differentiably basic* if for each differentiable function  $f: K \to \mathbf{R}$  there exist differentiable functions  $g: \mathbf{R} \to \mathbf{R}$  and  $h: \mathbf{R} \to \mathbf{R}$  such that f(x,y) = g(x) + h(y) for each point  $(x,y) \in K$ .

- 13. (a) (b) (c) Solve the analogues of problem 6 for differentiably basic sets.
- 14. (a) The graph V is differentiably basic.
- (b)  $W := (V (2,0)) \cup (V + (2,0))$  is not differentiably basic.
- (c) The broken line whose consecutive vertices are (-2,0), (-1,1), (0,0), (1,1) and (2,0) is not differentiably basic. (Note that it is continuously basic).
- (d) The completed array  $\{([\frac{n+1}{2}]^{-1/2}, [\frac{n}{2}]^{-1/2})\}_{n=2}^{\infty} \cup \{(0,0)\}$  is not differentiably basic. (Note that it is also not continuously basic.)
- (e) The completed array  $\{(2^{-[\frac{n+1}{2}]}, 2^{-[\frac{n}{2}]})\}_{n=1}^{\infty} \cup \{(0,0)\}$  is differentiably basic. (Note that it is not continuously basic.)
- (f) (I. Shnurnikov) The cross  $K = [(-1, -2), (1, 2)] \cup [(-1, 1), (1, -1)]$  is not differentiably basic. (This assertion and Conjecture 15a imply that the property of being differentiably basic is not hereditary.)
- (g) If a graph is basically embeddable in the plane, then it is differentiably basically embeddable in the plane. (This is non-trivial because the plane contains graphs which are basic but not differentiably basic and vice versa.) [RZ06]
- 15.\*\* Conjectures. (a) (I. Shnurnikov) A completed array  $\{a_n\}_{n=1}^{\infty} \cup \{(0,0)\}$  is differentiably basic if and only if the sequence  $\frac{\sum\limits_{n=k}^{\infty}|a_n|}{|a_k|}$  is bounded.
  - (b) The subset  $\{(t^2, \frac{t^2}{(1+t)^2})\}_{t\in[-\frac{1}{2};\frac{1}{2}]}$  of the plane is not differentiably basic.

Hint. One can try to prove this analogously to 14f. Cf. [Vo81, Vo82].

(c) A piecewise-linear graph in  $\mathbf{R}^2$  is differentiably basic if and only if it does not contain arbitrary long arrays and for each two singular points a and b we have  $x(a) \neq x(b)$  and  $y(a) \neq y(b)$ . A point  $a \in K$  is singular if the intersection of K with each disk centered at a is not a rectilinear arc.

It would be interesting to find a criterion of being differentiably basic for closed bounded subsets of the plane. Apparently a simple-to-state criterion (analogous to the Sternfeld criterion) does not exist. Another interesting question: is there a continuous map  $[0;1] \to \mathbb{R}^2$  whose image is differentiably basic but not basic?