BASIC EMBEDDINGS AND HILBERT'S 13TH PROBLEM [1]

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Abstract. This note is purely expository. In the course of the Kolmogorov-Arnold solution of Hilbert's 13th problem on superpositions there appeared the notion of *basic embedding*. A subset K of $\mathbf{R}^2$ is *basic* if for each continuous function $f\colon K\to\mathbf{R}$ there exist continuous functions $g,h\colon\mathbf{R}\to\mathbf{R}$ such that $f(x,y)=g(x)+h(y)$ for each point $(x,y)\in K$. We present descriptions of basic subsets of the plane (with a proof) and description of graphs basically embeddable into the plane (solutions of Arnold's and Sternfeld's problems). We present some results and open problems on the smooth version of the property of being basic. This note is accessible to undergraduates and could be an interesting easy reading for mature mathematicians. The two sections can be read independently on each other.

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HILBERT'S 13TH PROBLEM AND BASIC EMBEDDINGS

Hilbert's 13th problem

Let us recall informally the concept of *superposition*. Suppose that there is a set of functions of several variables, including all variables considered as functions. Represent each of the functions as an element of a circuit with several entries and one exit. Then a *superposition* of functions of this set is a function that can be repsesented by a circuit constructed from given elements; the circuit should not contain oriented cycles.

For example, a polynomial $a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ is a superposition of the constant functions and the functions $f(x,y)=x+y$, $g(x,y)=xy$. It is clear that any elementary function can be represented as a superposition of functions of at most two variables. *Is it possible to represent each function of several arguments as a superposition of functions of at most two arguments?*

Since there is a 1–1 correspondence between a segment and a square, any function of three and more variables is superposition of (in general, discontinuous) functions of two variables. So the above question is only interesting for continuous functions. [3] From now on we assume all functions to be continuous, unless the contrary is explicitly specified.

Hilbert's 13th problem. *Can the equation $x^7+ax^3+bx^2+cx+1=0$ of degree seven be solved without using functions of three variables?*

This question was answered affirmatively in 1957 by Kolmogorov and Arnold. They proved that any continuous function of n variables defined on a compact subset of $\mathbf{R}^n$ can be represented as a superposition of continuous functions of one variable and addition. For an exposition accessible to undergraduates see [Ar58]. See also [Vi04].

Basic embeddings into higher-dimensional spaces [4]

Ostrand extended the Kolmogorov-Arnold Theorem this theorem to arbitrary n-dimensional compacta [St89]. It is in the Kolmogorov-Arnold-Ostrand papers that the notion of basic subset

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[1] This is an English version of the paper in Russian under the same title. The English version has much shorter first section (which corresponds to two sections in Russian version), but contains solutions of problems 14a and 16c from the third section. Whenever possible I give references to surveys not to original papers. I would like to acknowledge V.I.Arnold, Yu.M. Burman, I.N. Shnurnikov, A.R. Safin, S.M. Voronin and M. Vyaliy for useful discussions, and M. Vyaliy for preparation of figures.

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[3] Denote by

$$|x,y|=|(x_1,\dots,x_n),\ (y_1,\dots,y_n)|=\sqrt{(x_1-y_1)^2+\dots+(x_n-y_n)^2}$$

the ordinary distance between points $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ of $\mathbf{R}^n$. Let K be a subset of $\mathbf{R}^n$. A function $f:K\to\mathbf{R}$ is called *continuous* if for each point $x_0\in K$ and number $\varepsilon>0$ there exists a number $\delta>0$ such that for each point $x\in K$ if $|x,x_0|<\delta$, then $|f(x)-f(x_0)|<\varepsilon$. E. g. the function $f(x_1,x_2)=\sqrt{x_1^2+x_2^2}$ is continuous on the plane, whereas the function $f(x_1,x_2)$ equal to the integer part of x_1+x_2 is not.

[4] This subsection is not used in the sequel and so can be omitted.

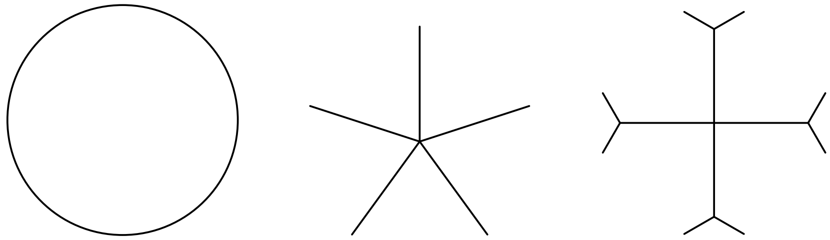


Figure 1:

appeared for the first time. It was explicitly introduced by Sternfeld [St89]. A subset $K \subset \mathbf{R}^m$ is *basic* if for each continuous function $f : K \to \mathbf{R}$ there exist continuous functions $g_1, \dots, g_m : \mathbf{R} \to \mathbf{R}$ such that $f(x_1, \dots, x_m) = g_1(x_1) + \dots + g_m(x_m)$ for each point $(x_1, \dots, x_m) \in K$.

Theorem 1. [St89] *Any n-dimensional compactum is basically embeddable into $\mathbf{R}^{2n+1}$ and, for $n > 1$, is not basically embeddable into $\mathbf{R}^{2n}$.*

It is interesting to compare this theorem with the Nöbeling-Menger-Pontryagin theorem on embeddability of any n-dimensional compact space into $\mathbf{R}^{2n+1}$ and the example of an n-dimensional polyhedron non-embeddable into $\mathbf{R}^{2n}$.

Obviously, K is basically embeddable into $\mathbf{R}$ if and only if K is topologically embeddable into $\mathbf{R}$. It follows from Theorem 1 that a compactum K is basically embeddable into $\mathbf{R}^m$ for $m > 2$ if and only if $\dim K < m/2$. Thus, the only remaining case is $m = 2$ (Sternfeld's problem).

Basic embeddings into the plane

A subset K of $\mathbf{R}^2$ is *basic* if for each continuous function $f \colon K \to \mathbf{R}$ there exist continuous functions $g, h \colon \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

Let us present the characterization of arcwise connected compacta basically embeddable into the plane [Sk95] (this is a partial solution of Sternfeld's problem). We formulate the criterion first for graphs and then for the general case. A conjecture on embeddability of (not necessarily arcwise connected) connected compacta into the plane can be found in [Sk95]. Compacta used in the statements are defined after the statements.

Theorem 2. [Sk95] *A finite graph K is basically embeddable into the plane if and only if any of the following two equivalent conditions holds:*

(a) K does not contain subgraphs homeomorphic to S, C_1, C_2 (fig. 1), that is, a circle, a five-point star, and a cross with branched endpoints;

(b) K is contained in one of the graphs R_n, $n = 1, 2, 3, \dots$ (fig. 2).

Let F_1 be a triod. The graph F_{n+1} is obtained from F_n by branching its endpoints (fig. 2). The graph R_n is obtained from F_n by by adding a hanging edge to each non-hanging vertex.

Theorem 3. [Sk95] *An arcwise-connected compactum K is basically embeddable into the plane if and only if it is locally connected (i.e., is a Peano continuum) and any of the two following (equivalent) conditions hold:*

(1) K does not contain S^1, C_2, C_4, B as subcompacta and contains only finitely many subcontinua F_n, H_n (fig. 1,2,3);

(2) K does not contain any of the continua $S^1, C_1, C_2, C_3, B, F, H_+, H_-, h_+, h_-$ (fig. 1,3,4).

Let $I = [0; 1]$. A sequence of sets is called a *null-sequence* if their diameters tend to zero. Define

• H_n to be the union of I with a null-sequence of triods having endpoints attached to I at points $3^{-l_1} + \dots + 3^{-l_s}$, where $s \le n$ and $0 < l_1 < \dots < l_s$ are integers;

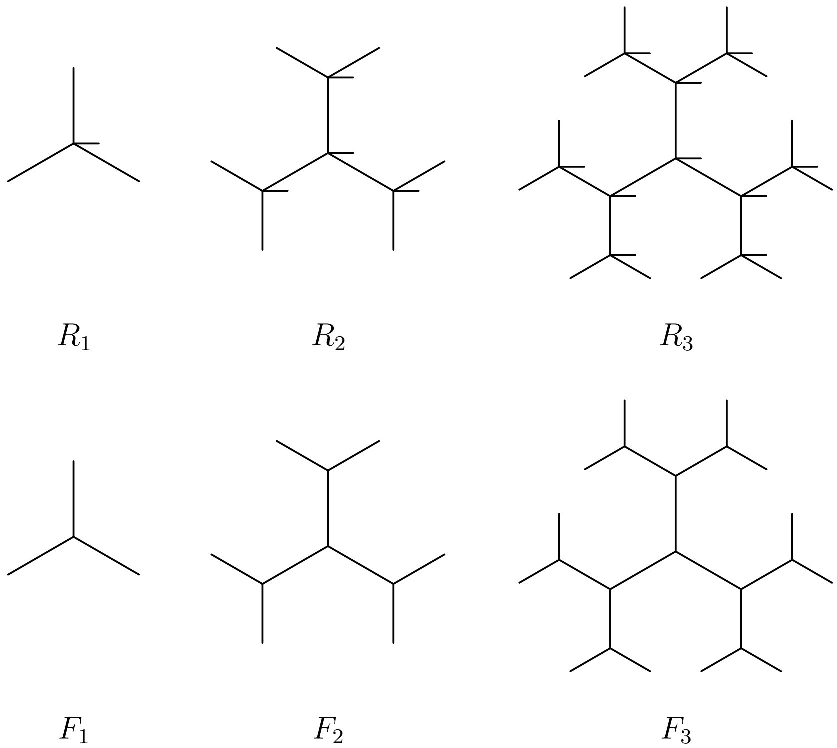


Figure 2:

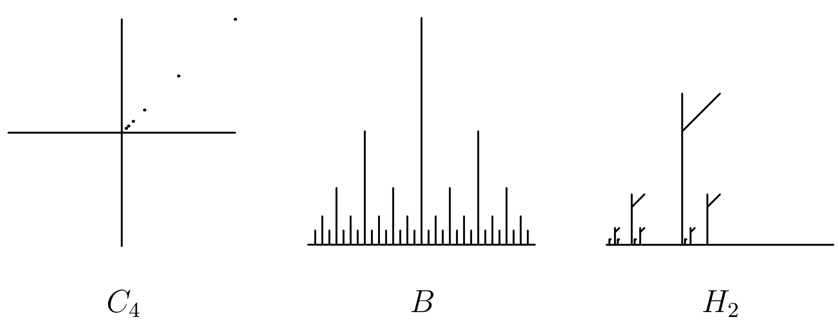
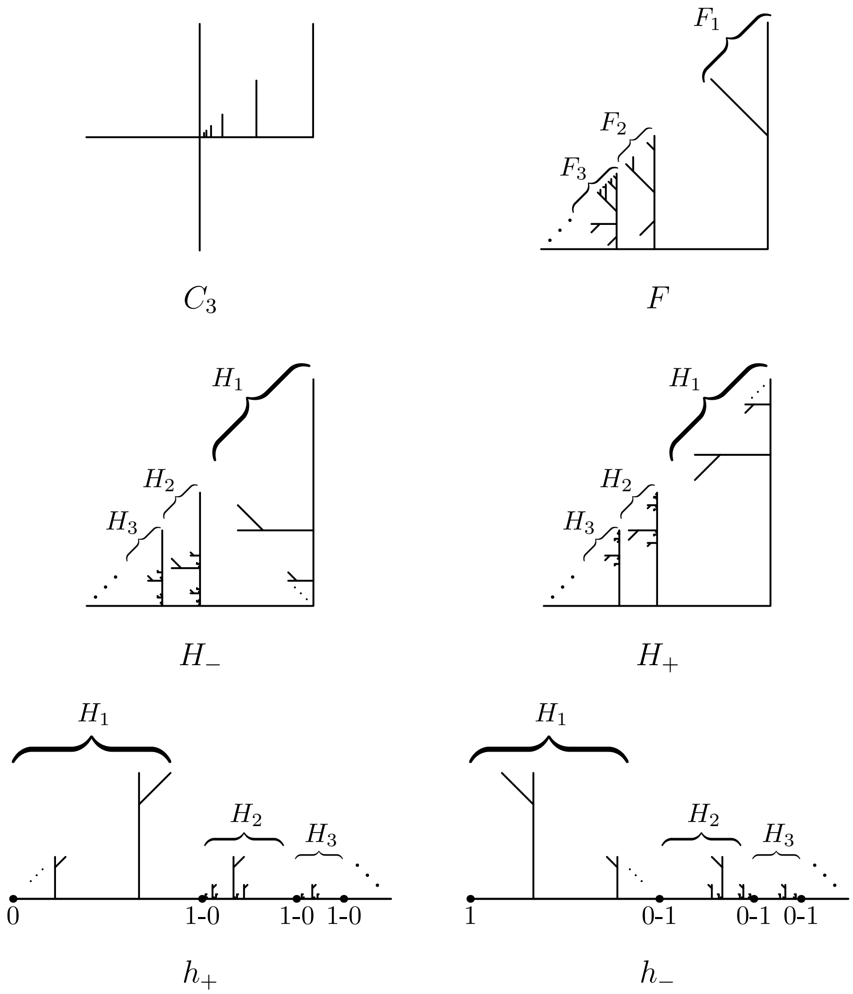


Figure 3:



F1
F2
F3
C3
F
H1
H2
H3
H−
H1
H2
H3
H+
H1
H2
H3
0
1-0
1-0
1-0
h+
H1
H2
H3
1
0-1
0-1
0-1
h−

Figure 4:

• C_3 to be a cross with a null-sequence of arcs attached to one of its branches and converging to its center;

• C_4 to be a cross with a sequence of points converging to its center;

• B to be the union of the arc I and a null-sequence of arcs attached to $(0;1)$ by their endpoints at rational points;

• F to be the union of I with a null-sequence of sets F_n each having an endpoint attached to the point $1/n \in I$;

• H_+ (H_-) to be the union of I with a null-sequence of continua H_n connected to the points $1/n \in I$ by arcs that intersect H_n at the points $1 \in I \subset H_n$ ($0 \in I \subset H_{n-1}$, respectively);

• h_+ (h_-) to be obtained from a null-sequence of continua H_n by pasting together the points $1 \in I \subset H_n$ and $0 \in I \subset H_{n-1}$ ($0 \in I \subset H_n$ and $1 \in I \subset H_{n-1}$, respectively).

An embedding $K \subset X \times Y$ is *basic* if for any continuous function $f : K \to \mathbf{R}$ there exist continuous functions $g : X \to \mathbf{R}$, $h : Y \to \mathbf{R}$ such that $f(x,y) = g(x) + h(y)$ for any point $(x,y) \in K$.

Denote by T_n an n-od, i.e., an n-pointed star. A vertex of a graph K is called *horrible* if its degree is greater than 4 and *awful* if its degree is equal to 4 and it is not an endpoint of a hanging edge. The *defect* of a graph K is the sum $\delta(K) = (deg A_1 - 2) + \dots + (deg A_k - 2)$, where $A_1, \dots, A_k$ are all the horrible and awful vertices of K.

Theorem 4. [Ku99] *A finite graph K admits a basic embedding $K \subset \mathbf{R} \times T_n$ if and only if K is a tree and either $\delta(K) < n$ or $\delta(K) = n$ and K has a horrible vertex with a hanging edge.*

BASIC PLANAR SETS

The material is presented as a sequence of problems, which is peculiar not only to Zen monasteries but also to elite mathematical education (at least in Russia). Difficult problems are marked by a star, and unsolved problems by two stars. If the statement of a problem is an assertion, then it is required to prove this assertion.

Discontinuously basic subsets.

1. (a) Is it true that for any four numbers $f_{11}, f_{12}, f_{21}, f_{22}$ there exist four numbers g_1, g_2, h_1, h_2 such that $f_{ij} = g_i + h_j$ for each $i, j = 1, 2$?

(b) Andrey Nikolaevich and Vladimir Igorevich play the 'Dare you to decompose!' game. Some cells of chessboard are marked. A. N. writes numbers in the marked cells as he wishes. V. I. looks at the written numbers and chooses (as he wishes) 16 numbers $a_1, \dots, a_8, b_1, \dots, b_8$ as 'weights' of the columns and the lines. If each number in a marked cell turns out to be equal to the sum of weights of the line and the row (of the cell), then V. I. wins, and in the opposite case (i.e., when the number in at least one marked cell is not equal to the sum of weights of the line and the row) A. N. wins.

Prove that V. I. can win no matter how A. N. plays if and only if there does not exist a closed route of a rook starting and turning only at marked cells (the route is not required to pass through each marked cell).

Let $\mathbf{R}^2$ be the plane with a fixed coordinate system. Let $x(a)$ and $y(a)$ be the coordinates of a point $a \in \mathbf{R}^2$. An ordered set (either finite or infinite) $\{a_1, \dots, a_n, \dots\} \subset \mathbf{R}^2$ is called an *array* if for each i we have $a_i \neq a_{i+1}$ and $x(a_i) = x(a_{i+1})$ for even i and $y(a_i) = y(a_{i+1})$ for odd i. It is not assumed that points of an array are distinct. An array is called *closed* if $a_1 = a_{2l+1}$.

2. Consider a closed array $\{a_1, \dots, a_n = a_1\}$. A *decomposition* for such an array is an assignment of numbers at the projections of the points of the array on the x-axis and on the y-axis. Is it possible to put numbers $f_1, \dots, f_n \in \mathbf{R}$, where $f_1 = f_n$, at the points of the array so that for each decomposition there exists an f_i that is not equal to the sum of the two numbers at $x(a_i)$ and $y(a_i)$?

A subset $K \subset \mathbf{R}^2$ is called *discontinuously basic* if for each function $f : K \to \mathbf{R}$ there exist functions $g, h : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

3. (a) The segment $K = 0 \times [0; 1] \subset \mathbf{R}^2$ is discontinuously basic.

(b) The cross $K = 0 \times [-1; 1] \cup [-1; 1] \times 0 \subset \mathbf{R}^2$ is discontinuously basic.

(c) *A criterion for a subset of the plane to be discontinuously basic.* A subset of the plane is discontinuously basic if and only if it does not contain any closed arrays.

4.** Given a set of marked unit cubes in the cube $8 \times 8 \times 8$, how can we see who wins in the 3D analogue of the 'Dare you to decompose!' game? In this analogue V. I. tries to choose 24 numbers $a_1, \dots, a_8, b_1, \dots, b_8, c_1, \dots, c_8$ so that the number at the unit cube (i, j, k) would be equal to the sum $a_i + b_j + c_k$ of the three weights.

5.** (a) Define discontinuous basic subsets of the 3-space. Discover and prove the 3D analogue of the above criterion.

(b) The same for higher-dimensional case.

Solutions.

1. (a) It is not true. If $f_{ij} = g_i + h_j$ for each $i, j = 1, 2$, then $f_{11} + f_{22} = f_{12} + f_{21}$, but this is false for some numbers f_{ij}.

(b) The statement 'only if' follows from the problem 2. Let us prove the 'if' part by induction on the number of the marked cells. If only one cell is marked then we are done. Let K be the set of centres of the marked cells. The set $E(K)$ is defined in the following subsection after Problem 9. The set K does not contain any closed array, therefore $\#E(K) < \#K$. So by the induction hypothesis V. I. can win for $E(K)$. Each cell from $K - E(K)$ is the only marked cell on its line or column, thus V. I. can choose the remaining weights for K.

2. Yes, it is. If every f_i is equal to the sum of two numbers at $x(a_i)$ and $y(a_i)$, then $f_1 - f_2 + f_3 - \dots - f_{n-1} = 0$, but this is false for some numbers f_i.

3. (a) Set $h(y) = f(0, y)$ and $g(x) = 0$.

(b) Set $g(x) = f(x, 0)$ and $h(y) = f(0, y) - f(0, 0)$.

(c) The statement 'only if' follows from the problem 2. Let us prove the 'if' part. Consider a function $f : K \to \mathbf{R}$. Our aim is to construct functions g and h so that $f(x, y) = g(x) + h(y)$. Two points $a, b \in K$ are called *equivalent* if there is an array $\{a = a_1, \dots, a_n = b\} \subset K$. Now take an equivalence class $K_1 \subset K$. Define function $g : x(K_1) \to \mathbf{R}$ and $h : y(K_1) \to \mathbf{R}$ in the following way. Take any point $a_1 \in K_1$ and set $g(x(a_1)) = f(a_1)$ and $h(y(a_1)) = 0$. If $\{a_1, a_2, \dots, a_{2l}\}$ is an array, then set

$$h(y(a_{2l})) := f(a_{2l}) - f(a_{2l-1}) + \dots - f(a_1) \quad \text{and} \quad g(x(a_{2l})) := f(a_{2l-1}) - f(a_{2l-2}) + \dots + f(a_1).$$

If $\{a_1, a_2, \dots, a_{2l+1}\}$ is an array, then set $g(x(a_{2l+1})) := f(a_{2l+1}) - f(a_{2l}) + \dots + f(a_1)$ ($h(y(a_{2l+1}))$ is already defined). Make this construction for each equivalence class. Then set $g = 0$ and $h = 0$ at all other points of $\mathbf{R}$.

Continuously basic subsets.

A subset $K \subset \mathbf{R}^2$ is called *(continuously) basic* if for each continuous function $f : K \to \mathbf{R}$ there exist continuous functions $g, h : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

The Arnold problem. *Which subsets of the plane are basic?* [Ar58]

In order to approach a solution consider some examples.

6. (a) A closed array is not basic.

(b) The segment $K = 0 \times [0; 1] \subset \mathbf{R}^2$ is basic.

(c) The cross $K = 0 \times [-1; 1] \cup [-1; 1] \times 0 \subset \mathbf{R}^2$ is basic.

(d) The graph V of the function $y = |x|$, $x \in [-1; 1]$ is basic.

A sequence of points $\{a_1, \dots, a_n, \dots\} \subset \mathbf{R}^2$ *converges to a point* $a \in \mathbf{R}^2$ if for each $\varepsilon > 0$ there exists an integer N such that for each $i > N$ we have $|a_i, a| < \varepsilon$.

7. (a) If a subset of the plane is basic, then it is discontinuously basic.

(b) A *completed array* is the union of a point $a_0 \in \mathbf{R}^2$ with an infinite array $\{a_1, \dots, a_n, \dots\} \subset \mathbf{R}^2$ of distinct points which converges to the point a_0. Prove that any completed array is not basic. (Note that it is discontinuously basic).

(c) Let $[a, b]$ be the rectilinear arc which connects points a and b. Prove that the cross $K = [(-1,-2),(1,2)] \cup [(-1,1),(1,-1)]$ is not basic.

(d) Let $m_{ij} = 2 - 3 \cdot 2^{-i} + j \cdot 2^{-2i}$. Consider the set of points $(m_{i,2l}, m_{i,2l})$ and $(m_{i,2l}, m_{i,2l-2})$, where i varies from 1 to ∞ and $l = 1, 2, 3, \dots, 2^{i-1}$. Prove that this subset of the plane does not contain any infinite arrays but contains arbitrary long arrays.

(e) The union of the set from the previous problem and the point $(2,2)$ is not basic.

A subset $K \subset \mathbf{R}^2$ of the plane is *closed*, if for each sequence $a_i \in K$ converging to a point a this point belongs to K.

8. A subset $K \subset \mathbf{R}^2$ of the plane is closed if and only if for each point $a \notin K$ there exists $\varepsilon > 0$ such that if for a point b of the plane we have $|a, b| \le \varepsilon$, then b does not belong to K.

The Sternfeld criterion for being a basic subset. *A closed bounded subset $K \subset \mathbf{R}^2$ of the plane is basic if and only if K does not contain arbitrary long arrays.*

9. (a) The criterion is false without the assumption that K closed.

(b) The criterion is false without the assumption that K bounded.

(c)** Find a criterion of being a basic subset for closed (but unbounded) subsets of the plane.

Suppose that K is a subset of $\mathbf{R}^2$. For every point $v \in K$ consider the pair of lines passing through v and parallel to the x-axis and the y-axis. If one of these two lines intersects K only at point v, we colour v in white. Define $E(K)$ as the set of noncoloured points of K:

$$E(K) = \{v \in K : \ |K \cap (x = x(v))| \ge 2 \text{ and } |K \cap (y = y(v))| \ge 2\}.$$

Let $E^2(K) = E(E(K))$, $E^3(K) = E(E(E(K)))$ etc.

10. A subset K of the plane does not contain arbitrary long arrays if and only if $E^n(K) = \emptyset$ for some n.

12. (a)* Give an elementary proof that if K is a closed bounded subset of $\mathbf{R}^2$ and $E(K) = \emptyset$, then K is basic [Mi09].

Hint. It can be proven that for piecewise-linear maps f there is a decomposition $f(x,y) = g(x) + h(y)$ with $|g| + |h| < 5|f|$.

(b)* Prove the 'if' part of the criterion without using the functional spaces as below.

Hint. Same as above with $|g| + |h| < C_n|f|$, where C_n depends only on that n for which $E^n(K) = \emptyset$.

11. A subset $K \subset \mathbf{R}^3$ is called *(continuously) basic* if for each continuous function $f : K \to \mathbf{R}$ there exist continuous functions $g, h, l : \mathbf{R} \to \mathbf{R}$ such that $f(x,y,z) = g(x) + h(y) + l(z)$ for each point $(x,y,z) \in K$.

(a) The 'hedgehog' $0 \times 0 \times [-1;1] \cup 0 \times [-1;1] \times 0 \cup [-1;1] \times 0 \times 0 \subset \mathbf{R}^3$ is basic.

(b) The set of 4 points $(0,0,0)$; $(1,1,0)$; $(0,1,1)$; $(1,0,1)$ is basic. (But $E^n(K) \ne \emptyset$ for each n, see below.)

(c)* Define $E(K)$ analogously to the above, only instead of lines use planes orthogonal to the axes:

$$E(K) = \{v \in K : \ |K \cap (x = x(v))| \ge 2, \ |K \cap (y = y(v))| \ge 2 \text{ and } |K \cap (z = z(v))| \ge 2\}.$$

Let K be a closed bounded subset of $\mathbf{R}^3$. Prove that if $E^n(K) = \emptyset$ for some n, then K is basic [St89, Lemma 23.ii].

Solutions.

6. (a) If an array $A = \{a_1, \dots, a_{2l+1}\}$ is basic, then $f(a_1) - f(a_2) + \dots + f(a_{n-2}) - f(a_{2l}) = 0$. But this is false for some functions f. Cf. problem 2.

(b),(c) Analogously to problems 3a,3b.

(d) Take $h(y) = 0$ and $g(x) = f(x, y)$.

7. (a) If the subset is not discontinuously basic, then it contains a closed array. Hence the statement follows by extension of f on the subset and using problem 6a.

(b) Define function f by $f(a_n) = \frac{(-1)^n}{n}$. Suppose that $f(x, y) = g(x) + h(y)$ for some g and h. Then

$$f(a_1) - f(a_2) + f(a_3) - f(a_4) + \dots - f(a_{2l}) = h(y(a_1)) - h(y(a_{2l})).$$

Since $\lim_{l\to\infty} h(y_{2l})$ exists and equals to $h(y(a_0))$, it follows that $\sum_{i=1}^{2l}(-1)^i f(a_i)$ converges when $l \to \infty$, which is a contradiction.

(c) The cross contains a completed array

$$a_{4k+1} = (-2^{-2k}, 2^{-2k}),\ a_{4k+2} = (2^{-2k-1}, 2^{-2k}),\ a_{4k+3} = (2^{-2k-1}, -2^{-2k-1}),\ a_{4k+4} = (-2^{-2k-2}, -2^{-2k-1}).$$

Define a function f on this array using problem 7.b and then extend it (e.g. piecewise linearly) to the cross. Then there are no functions g and h such that $f(x, y) = g(x) + h(y)$.

(d) For every i the set $(m_{i,2l}, m_{i,2l})_{l=1}^{2^{i-1}} \cup (m_{i,2l}, m_{i,2l-2})_{l=1}^{2^{i-1}}$ is an array of 2^i points.

(e) Define a function f by

$$f((m_{i,2l}, m_{i,2l})) := 2^{-i} \quad \text{and} \quad f(m_{i,2l}, m_{i,2l-2}) := -2^{-i}.$$

If $f(x, y) = g(x) + h(y)$ for some g and h, then for every i using array of points $(m_{i,2l}, m_{i,2l})$ and $(m_{i,2l}, m_{i,2l-2})$, where $l = 1, 2, 3, \dots 2^{i-1}$, we obtain $h(2 - \frac{3}{2^i}) - h(2 - \frac{2}{2^i}) = 1$. This contradicts to the continuity of h.

8. Let us prove the 'only if' part. Let K be a closed subset of the plane. Suppose that for some point $a = (x, y) \notin K$ and for each $\varepsilon = \frac{1}{n} > 0$ there exists a point $a_n \in K$ (at least one) such that $|a, a_n| \le \frac{1}{n}$. The sequence of points $a_n \in K$ converges to the point a, thus $a \in K$. Contradiction.

Now let us prove the 'if' part. Suppose that a sequence a_n converges to a point a and the point $a = (x, y)$ is not in K. There exists $\varepsilon > 0$ such that for every point $a_n \in K$ the distance $|a, a_n| > \varepsilon$. This is a contradiction.

9. (a) Any infinite array A not containing closed arrays and converging to a point $a \notin A$ is basic. This follows because each function defined on A is continuous.

(b) A counterexample is $\{(k, k)\}_{k=1}^{\infty} \cup \{(k, k-1)\}_{k=1}^{\infty}$.

10. Let us prove the 'only if' part. Suppose that $E^n(K) \ne \emptyset$ for each n. For each n take a point $a_0 \in E^n(K)$. Then there exist points $a_{-1}, a_1 \in E^{n-1}(K)$ such that $x(a_{-1}) = x(a_0)$ and $y(a_1) = y(a_0)$. Analogously there exist points $a_{-2}, a_2 \in E^{n-2}(K)$ such that $\{a_{-2}, a_{-1}, a_0, a_1, a_2\}$ is an array. Analogously we construct an array of $2n + 1$ points in K, which is a contradiction.

Let us prove the 'if' part. Suppose that K contains an array of $2n + 1$ points $\{a_{-n}, \dots, a_0, \dots, a_n\}$. Then there is an array of $2n - 1$ points $\{a_{-n+1}, \dots, a_{n-1}\}$ in $E(K)$. Analogously $a_0 \in E^n(K)$. Thus if $E^n(K) = \emptyset$, then K does not contain an array of $2n + 1$ points.

11. (a) For each functuion $f : K \to \mathbf{R}$ on K define $g(x) := f(x, 0, 0)$, $h(y) := f(0, y, 0) - f(0, 0, 0)$ and $l(z) := f(0, 0, z) - f(0, 0, 0)$.

(b) Set $g(0) = f(0, 0, 0)$, $h(0) = 0$, $l(0) = 0$,

$$2g(1) = f(0,0,0) + f(1,1,0) + f(1,0,1) - f(0,1,1), \quad 2h(1) = -f(0,0,0) + f(1,1,0) - f(1,0,1) + f(0,1,1)$$

$$\text{and} \quad 2l(1) = -f(0,0,0) - f(1,1,0) + f(1,0,1) + f(0,1,1).$$

Proof of the criterion for being a basic subset.

Let K be a closed bounded subset of the plane. It is known that each continuous function $f : K \to \mathbf{R}$ is bounded. A function $f : K \in \mathbf{R}$ is called *bounded*, if there exists a number M such that $|f(x)| < M$ for every $x \in K$. For a bounded function $G : K \to \mathbf{R}$ denote $|G| := \sup_{x\in K} |G(x)|$.

Beginning of the proof of the 'only if' part of the criterion. Assume to the contrary that K contains arbitrary long arrays and is basic. Choosing subsequences we may assume that points of each array are distinct. Therefore for each n there is an array $\{a_1^n,\dots,a_{2n+5}^n\}$ of $2n+5$ distinct points in K.

Then there exists continuous function

$$f_n : K \to \mathbf{R} \quad \text{such that} \quad f_n(a_i^n) = (-1)^i \quad \text{and} \quad |f_n(x)| \le 1 \quad \text{for each} \quad x \in K.$$

(Indeed, first define such a continuous function $f : \mathbf{R}^2 \to \mathbf{R}$. Denote $s = \min_{i<j} |a_i, a_j|$. Take n disks with centers a_i and radii $\frac{s}{3}$. Outside of these disks set $f = 0$. Inside the i-th disk take f to be $(-1)^i$ in the center a_i, 0 on the boundary and extend it linearly in the distance to a_i. Then restrict f to $K \subset \mathbf{R}^2$.)

Define integers s_n and functions $F_n : K \to \mathbf{R}$ inductively as follows. Set $s_0 = 1$ and $F_0 = 0$. Suppose now that F_{n-1} and s_{n-1} are defined. If F_{n-1} is not representable as $G_{n-1}(x) + H_{n-1}(y)$, then we are done. If it is representable in this way, then take

$$s_n > s_{n-1}!(|G_{n-1}| + n) \qquad \text{and} \qquad F_n = F_{n-1} + \frac{f_{s_n}}{s_{n-1}!}$$

It remains to prove that if we can construct in this way an infinite number of s_n and F_n, then the function

$$F = \lim_{n\to\infty} F_n = \sum_{n=1}^{\infty} \frac{f_{s_n}}{s_{n-1}!}$$

is not representable as $G(x) + H(y)$.

Assume to the contrary that $F(x,y) = G(x) + H(y)$ for some G and H. It suffices to prove that $|G| > n$ for each n. For this it suffices to prove that $s_{n-1}!|G - G_{n-1}| > s_n$: then we would have

$$|G| + |G_{n-1}| \ge |G - G_{n-1}| > \frac{s_n}{s_{n-1}!} > |G_{n-1}| + n.$$

Lemma. *Let $m \ge 4$,*

- $K = \{a_1,\dots,a_{2m+5}\}$ *be an array of* $2m+5$ *distinct points,*
- $f(a_1),\dots,f(a_{2m+5})$ *numbers such that* $|(-1)^i - f(a_i)| \le 1/m$,
- $g(x(a_i)), h(y(a_i))$, $i = 1,\dots,2m+5$, *numbers such that* $f_i = g(x(a_i)) + h(y(a_i))$ *for each* i.

Then $\max_i |g(x(a_i))| > n$.

Proof. We may assume that $a_1a_2 \| Ox$. Then

$$|(f(a_1) - f(a_2) + f(a_3) - f(a_4) + \dots - f(a_{2m+4}) - (2m+4)| \le \frac{2m+4}{m} \le 3.$$

Therefore $g(x(a_1)) - g(x(a_{2m+4})) \ge (2m+4) - 3 > 2m$. This implies the required inequality. □

Completion of the proof of the 'only if' part of the criterion. We have

$$F - F_n = F - F_{n-1} - \frac{f_{s_n}}{s_{n-1}!} = \frac{s_{n-1}!(F - F_{n-1}) - f_{s_n}}{s_{n-1}!}.$$

Apply the Lemma to

$$m = s_n, \quad a_i = a_i^{s_n}, \quad f = s_{n-1}!(F - F_{n-1}), \quad g = s_{n-1}!(G(x) - G_{n-1}(x)), \quad h = s_{n-1}!(H(y) - H_{n-1}(y)).$$

This is possible because $f(x,y) = g(x) + h(y)$ and (since $s_n - 1 > s_{n-1}$ for $n > 2$)

$$|f - f_{s_n}| = s_{n-1}!|F - F_n| < \frac{1}{(s_n-1)\cdot s_n} \sum_{k=0}^{\infty} \frac{1}{(s_n+1)\cdot \dots \cdot s_{n+k}} < \frac{1}{(s_n-1)\cdot s_n} \sum_{k=0}^{\infty} \frac{1}{2^k} < \frac{1}{s_n}.$$

By Lemma we obtain $s_{n-1}!|G - G_{n-1}| > s_n$. $\square$

Proof of the criterion. [5] The proof is based on a reformulation of the property of being a basic subset in terms of *bounded linear operators* in *Banach functional spaces.* Denote by $C(X)$ the space of continuous functions on X with the norm $|f| = \sup\{|f(x)| \ : \ x \in X\}$. In this proof denote by $pr_x(a)$ and $pr_y(a)$ the projections of a point $a \in K$ on the coordinate axes.

For $K \subset I^2 := [0;1] \times [0;1]$ define a map (*linear superposition operator*)

$$\phi\colon C(I) \oplus C(I) \to C(K) \quad \text{by} \quad \phi(g,h)(x,y) := g(x) + h(y).$$

Clearly, the subset $K \subset I^2$ is basic if and only if ϕ is surjective, or equivalently, epimorphic.

Denote by $C^*(X)$ the space of *bounded linear functions* $C(X) \to \mathbf{R}$ with the norm $|\mu| = \sup\{|\mu(f)| \ : \ f \in C(X),\ |f| = 1\}$. For a subset $K \subset I^2$ define a map (*dual linear superposition operator*)

$$\phi^*\colon C^*(K) \to C^*(I) \oplus C^*(I) \quad \text{by} \quad \phi^*\mu(g,h) := (\mu(g \circ pr_x), \mu(h \circ pr_y)).$$

Since $|\phi^*\mu| \le 2|\mu|$, it follows that ϕ^* is bounded. By duality, ϕ is epimorphic if and only if ϕ^* is monomorphic. [6]

It is clear that ϕ^* is monomorphic if and only if

(*) there exists $\varepsilon > 0$ such that $|\phi^*\mu| > \varepsilon|\mu|$ for each unzero $\mu \in C^*(K)$.

We leave as an excercise the proof that (*) implies the abcense of arbitrarily large arrows. (This proves the 'only if' part of the criterion, for which we already have an elementary proof.)

So it remains to prove that $E^n(K) = \emptyset$ implies the condition (*). We present the proof for $n \in \{1,2\}$. The proof for arbitrary n is analogous. We use the following non-trivial fact: $C^*(X)$ *is the space of σ-additive regular real valued Borel measures on* X (in the sequel we call them simply 'measures'). We have

$$\phi^*\mu = (\mu_x, \mu_y), \quad \text{where} \quad \mu_x(U) = \mu(pr_x^{-1}U) \quad \text{and} \quad \mu_y(U) = \mu(pr_y^{-1}U) \quad \text{for each Borel set} \quad U \subset I.$$

If $\mu = \mu^+ - \mu^-$ is the decomposition of a measure μ into its positive and negative parts, then $|\mu| = \bar\mu(X)$, where $\bar\mu = \mu^+ + \mu^-$ is the absolute value of μ.

Let D_x (D_y) be the set of points of K which are not shadowed by some other point of K in x-(y-) direction. Take any measure μ on K of the norm 1.

If $n = 1$, then

$$E(K) = \emptyset, \quad \text{then} \quad D_x \cup D_y = K, \quad \text{so} \quad 1 = \bar\mu(K) \le \bar\mu(D_x) + \bar\mu(D_y).$$

Therefore without loss of generality, $\bar\mu(D_x) \ge 1/2$. Since the projection onto the x-axis is injective over D_x, it follows that $|\mu_x| \ge 1/2$, thus the required assertion holds for $\varepsilon = \frac{1}{2}$.

If $n = 2$, then

$$E(E(K)) = \emptyset, \quad \text{hence} \quad D_x \cup D_y = K - E(K), \quad \text{so} \quad E(D_x \cup D_y) = \emptyset.$$

In the case when $\bar\mu(E(K)) < 3/4$ we have $\bar\mu(D_x \cup D_y) > 1/4$ and without loss of generality $\bar\mu(D_x) > 1/8$. Then as for $n = 1$ we have $|\mu_x| > 1/8$, thus (*) holds for $\varepsilon = \frac{1}{8}$.

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[5] This proof is not elementary, is not used in the sequel and could be omitted.

[6] We remark that ϕ^* can be injective but not monomorphic. In other words not only some linear relation on $\operatorname{im}\phi$ can force it to be strictly less than $C(K)$.
If an embedding $K \subset \mathbf{R}^2$ is basic, then we can prove that ϕ^* is monomorphic without use of ϕ as follows. Define a linear operator

$$\Psi\colon C^*(I) \oplus C^*(I) \to C^*(K) \quad \text{by} \quad \Psi(\mu_x, \mu_y)(f) = \mu_x(g) + \mu_y(h),$$

where $g, h \in C(I)$ are such that $g(0) = 0$ and $f(x,y) = g(x) + h(y)$ for $(x,y) \in K$. Clearly, $\Psi\phi^* = \mathrm{id}$ and Ψ is bounded, hence ϕ^* is monomorphic.

In the case when $\bar\mu(E(K)) \geq 3/4$ we have $\bar\mu(K - E(K)) \leq 1/4$. By the case $n = 1$ above without loss of generality $\bar\mu_x(pr_x(E(K))) \geq \bar\mu(E(K))/2$. Hence $|\mu_x| \geq \frac{1}{2}\cdot\frac{3}{4} - \frac{1}{4} = \frac{1}{8}$, thus (*) holds for $\varepsilon = \frac{1}{8}$. □

Smoothly basic subsets of the plane.

Let K be a subset of the plane $\mathbf{R}^2$. A function $f : K \to \mathbf{R}$ is called *differentiable* if for each point $z_0 \in K$ there exist a vector $a \in \mathbf{R}^2$ and infinitesimal function $\alpha : \mathbf{R}^2 \to \mathbf{R}$ such that for each point $z \in K$

$$f(z) = f(z_0) + a\cdot(z - z_0) + \alpha(z - z_0)|z, z_0|.$$

Here the dot denotes scalar product of vectors $a =: (f_x, f_y)$ and $z - z_0 =: (x, y)$, i.e. $a\cdot(z - z_0) = xf_x + yf_y$. A function $\alpha : \mathbf{R}^2 \to \mathbf{R}$ is *infinitesimal*, if for each number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for each point $(x, y) \in \mathbf{R}^2$

$$\text{if} \quad \sqrt{x^2 + y^2} < \delta, \quad \text{then} \quad |\alpha(x, y)| < \varepsilon.$$

Let V be the graph of the function $y = |x|$, where $x \in [-1; 1]$. A function $f : V \to \mathbf{R}$ is differentiable if and only if $f(x, |x|)$ is differentiable on the segments $[-1; 0]$ and $[0; 1]$.

A subset $K \subset \mathbf{R}^2$ of the plane is called *differentiably basic* if for each differentiable function $f : K \to \mathbf{R}$ there exist differentiable functions $g : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

13. (a) (b) (c) Solve the analogues of problem 6 for differentiably basic sets.

14. (a) The graph V is differentiably basic.

(b) $W := (V - (2, 0)) \cup (V + (2, 0))$ is not differentiably basic.

(c) The broken line whose consecutive vertices are $(-2, 0)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 0)$ is not differentiably basic. (Note that it is continuously basic).

(d) The completed array $\{([\frac{n+1}{2}]^{-1/2}, [\frac{n}{2}]^{-1/2})\}_{n=2}^{\infty} \cup \{(0, 0)\}$ is not differentiably basic. (Note that it is also not continuously basic.)

(e) The completed array $\{(2^{-[\frac{n+1}{2}]}, 2^{-[\frac{n}{2}]})\}_{n=1}^{\infty} \cup \{(0, 0)\}$ is differentiably basic. (Note that it is not continuously basic.)

(f) (I. Shnurnikov) The cross $K = [(-1, -2), (1, 2)] \cup [(-1, 1), (1, -1)]$ is not differentiably basic. (This assertion and Conjecture 15a imply that the property of being differentably basic is not hereditary.)

(g) If a graph is basically embeddable in the plane, then it is differentiably basically embeddable in the plane. (This is non-trivial because the plane contains graphs which are basic but not differentaibly basic and vice versa.) [RZ06]

15. Conjectures.** (a) (I. Shnurnikov) A completed array $\{a_n\}_{n=1}^{\infty} \cup \{(0, 0)\}$ is differentiably basic if and only if the sequence $\frac{\sum_{n=k}^{\infty}|a_n|}{|a_k|}$ is bounded.

(b) The subset $\{(t^2, \frac{t^2}{(1+t)^2})\}_{t\in[-\frac{1}{2};\frac{1}{2}]}$ of the plane is not differentiably basic.

Hint. One can try to prove this analogously to 14f. Cf. [Vo81, Vo82].

(c) A piecewise-linear graph in $\mathbf{R}^2$ is differentiably basic if and only if it does not contain arbitrary long arrays and for each two singular points a and b we have $x(a) \neq x(b)$ and $y(a) \neq y(b)$. A point $a \in K$ is *singular* if the intersection of K with each disk centered at a is not a rectilinear arc.

It would be interesting to find a criterion of being differentiably basic for closed bounded subsets of the plane. Apparently a simple-to-state criterion (analogous to the Sternfeld criterion) does not exist. Another interesting question: is there a continuous map $[0; 1] \to \mathbf{R}^2$ whose image is differentiably basic but not basic?

16. Let $r \geq 0$ be an integer and $K \in \mathbf{R}^2$ a subset. A function $f : K \to \mathbf{R}$ is called r *times differentiable* if for each point $z_0 \in K$ there exist a polynomial $\overline{f}(z) = \overline{f}(x,y)$ of degree at most r of 2 variables x and y and an infinitesimal function $\alpha : \mathbf{R}^2 \to \mathbf{R}$ such that $f(z) = \overline{f}(z - z_0) + \alpha(z - z_0)|z, z_0|^r$ for each point $z \in K$. (This definition differs from the one generally accepted.)

(a) Functions differentiable zero times are exactly continuous functions, and functions differentiable one time are exactly differentiable functions.

(b) For each positive integer r define the property of being an r times differentiably basic subset of the plane $\mathbf{R}^2$.

(c) For each integer $k \geq 0$ there is a subset of the plane which is r times differentiably basic for $r = 0, 1 \dots k$ but is not r times differentiably basic for each $r > k$.

(d)** Find a criterion for graphs in $\mathbf{R}^2$ to be r times differentiably basic.

Solutions.

13. (a), (b), (c) Analogously to problems 6(a), 3(a) and 3(b).

14. (a) Take a differentiable function $f : V \to \mathbf{R}$. Since f is differentiable at $(0,0)$, it follows that there exist $a, b \in \mathbf{R}$ such that

$$f(x, |x|) = f(0,0) + ax + b|x| + \alpha(x), \quad \text{where} \quad \alpha(x) = o(\sqrt{x^2 + |x|^2}) \quad \text{when} \quad x \to 0.$$

Take $h(y) := by$ and $g(x) := f(0,0) + ax + \alpha(x)$. Clearly, h is differentiable and g is differentiable outside 0. Since $\alpha(x) = o(x)$ when $x \to 0$, it follows that g is differentiable also at 0.

(b) See 16c for $k = 0$.

(c) Suppose the broken line is differentiably basic. The function $f(x,y) = xy$ is differentiable. We have $f(x,y) = g(x) + h(y)$, where both g and h are differentiable. Then

$$2-2d = f(1+d, 1-d) + f(1-d, 1-d) = g(1+d) + g(1-d) + 2h(1-d) = 2g(1) + 2h(1) - 2h'(1)d + o(d).$$

Hence $h'(1) = 1$. Analogously

$$2d-2 = f(-1+d, 1-d) + f(-1-d, 1-d) = g(-1+d) + g(-1-d) + 2h(1-d) = 2g(-1) + 2h(1) - 2h'(1)d + o(d).$$

Hence $h'(1) = -1$. A contradiction.

(d) Suppose that this completed array is differentiably basic. Set $a_n = ([\frac{n+1}{2}]^{-1/2}, [\frac{n}{2}]^{-1/2})$, $f(a_n) := \frac{(-1)^n}{n}$, $n = 2, 3, \dots$. If $f(x,y) = g(x) + h(y)$ for some functions $g(x)$ and $h(y)$, then the series $f(a_2) - f(a_3) + f(a_4) - \dots$ converges to $g(1) - g(0)$ (analogously to Problem 7b). This is a contradiction because the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges.

(e) Without loss of generality assume that $f(0,0) = 0$, then take $g(0) = 0$ and $h(0) = 0$. Set

$$h(2^{-k}) = f(2^{-(k+1)}, 2^{-k}) - f(2^{-(k+1)}, 2^{-(k+1)}) + f(2^{-(k+2)}, 2^{-(k+1)}) - \dots,$$

$$g(2^{-k}) = f(2^{-k}, 2^{-k}) - f(2^{-(k+1)}, 2^{-k}) + f(2^{-(k+1)}, 2^{-(k+1)}) - \dots,$$

where the right-hand sides are sums of alternating series. Now $g(x)$ and $h(y)$ may be extended to differentiable functions $\mathbf{R} \to \mathbf{R}$.

(f) Define

$$w(0) = w(4^{-i} + 4^{-3i}) = w(4^{-i}) = 0 \quad \text{and} \quad w(4^{-i} + 4^{-3i-1}) = 2^{3i} \quad \text{for} \quad i = 1, 2, 3, \dots.$$

Extend piecewise-linearly to obtain a function $w : [0;1] \to \mathbf{R}$. For $x \in [0;1]$ define $W(x)$ as the area under the graph of w on $[0;x]$. (This is well-defined because this area is finite.) Define $f(x,-x) = W(x)$ for $x \in [0;1]$ and $f(x,y) = 0$ on the rest of the cross.

Clearly, f is differentiable outside $(0,0)$. It is easy to check that f is differentiable at $(0,0)$.

Suppose that $f(x,y)=g(x)+h(y)$ for some differentiable functions g and h. Without loss of generality we assume that $g(0)=h(0)=0$. The function g is not differentiable at $x=1/4$ because for $0<d<\frac{1}{4}$ we have

$$g\left(\frac{1}{4}+d\right)-g\left(\frac{1}{4}\right)=W\left(\frac{1}{4}+d\right)-W\left(\frac{1}{4}\right)+W\left(\frac{1}{4^2}+\frac{d}{4}\right)-W\left(\frac{1}{4^2}\right)+\dots>$$

$$>W\left(\frac{1}{4^{k+1}}+\frac{d}{4^k}\right)-W\left(\frac{1}{4^{k+1}}\right)=\frac{2^{3k}\cdot 4^{-3k}}{2}\ge\frac{(4d)^{3/4}}{2}.$$

Here

• the first equality is proved using two infinite arrays starting at points $(\frac{1}{4}+d,-\frac{1}{4}-d)$ and $(\frac{1}{4},-\frac{1}{4})$ and converging to the point $(0,0)$;

• $k\ge 0$ is such that $4^{-2k}\ge 4d>4^{-2(k+1)}$;

• the first inequality follows because W is a non-decreasing function;

• the second inequality follows because $\frac{d}{4^k}>\frac{1}{4^{3(k+1)}}$;

• the second equality follows by definition of k.

(In the same way one can prove that g is not differentiable at $x=4^{-i}$ for each i.)

15. (a) Hints. For the 'only if' part use the idea of Problem 7b and prove that if $\sum\limits_{n=1}^{\infty}|a_n|=\infty$, then there is a sequence $b_n\to 0$ such that $\sum\limits_{n=1}^{\infty}|a_n|b_n=\infty$.

For the 'if' part we may assume that numbers $x(a_i)$ are distinct, numbers $y(a_i)$ are distinct, $x(a_{2i})=x(a_{2i+1})$, $y(a_{2i})=y(a_{2i-1})$. If $f(0,0)=0$, define

$$g(x(a_{2i})):=f(a_1)-f(a_2)+f(a_3)-\dots+f(a_{2i+1}),\quad g(0):=\sum_{i=1}^{\infty}(-1)^i f(a_i),$$

$$h(y(a_{2i})):=-f(a_1)+f(a_2)-f(a_3)-\dots+f(a_{2i-2})\quad\text{and}\quad h(0):=\sum_{i=1}^{\infty}(-1)^i f(a_i).$$

Prove that g and h are differentiable at 0.

16. (a) It is clear.

(b) A subset $K\subset\mathbf{R}^2$ is called r *times differentiably basic* if for each r times differentiable function $f:K\to\mathbf{R}$ there exist r times differentiable functions $g:\mathbf{R}\to\mathbf{R}$ and $h:\mathbf{R}\to\mathbf{R}$ such that $f(x,y)=g(x)+h(y)$ for each point $(x,y)\in K$.

(c) We can take the graph V_k of the function $y=|x|^k$, $x\in[-1;1]$ for k odd, and $W_{k+1}=(V_{k+1}-(2,0))\cup(V_{k+1}+(2,0))$ for k even.

Proof for k even. Let us prove that W_{k+1} is r times differentiably basic for each $0\le r\le k$. Given an r times differentiable function $f:W_{k+1}\to\mathbf{R}$, take functions $h(y)=0$ and $g(x)=f(x,|x-2\operatorname{sign}x|^{k+1})$. Clearly, h is r times differentiable and $f(x,y)=g(x)+h(y)$ for each $(x,y)\in W_{k+1}$. Since the function $p(t)=|t|^{k+1}$ is k times differentiable and $r\le k$, it follows that g is r times differentiable.

Let us prove that W_{k+1} is not r times differentiably basic for k even and each $k<r$. Define a function $f:W_{k+1}\to\mathbf{R}$ by $f(x,y)=y\operatorname{sign}x$. Clearly, f is r times differentiable. If W_{k+1} is r times differentiably basic, then there are r times differentiable functions g and h such that $f(x,y)=g(x)+h(y)$. For $t\in[-1;1]$ we have

$$g(\pm 2+t)+h(|t|^{k+1})=f(\pm 2+t,|t|^{k+1})=\pm|t|^{k+1}.$$

Since g is $(k+1)$ times differentiable and $k+1$ is odd, it follows that $h'(0)=+1$ and $h'(0)=-1$, which is a contradiction. □

Proof for k odd. First we prove that V_k is r times differentiably basic for each $0\le r\le k$. Take an r times differentiable function $f:V_k\to\mathbf{R}$. Since f is r times differentiable at $(0,0)$, it follows that there exist $\{a_{ij}\}_{i,j=0}^{r}\subset\mathbf{R}$ such that

$$a_{00}=f(0,0)\quad\text{and}\quad f(x,|x|^k)=\sum_{i,j=0}^{r}a_{ij}x^i|x|^{kj}+o([x^2+x^{2r}]^{r/2})\quad\text{when}\quad x\to 0.$$

Since

$$o([x^2+x^{2r}]^{r/2})=o_1(x^r),\quad \text{we have}\quad f(x,|x|^k)=a_{00}+a_{01}|x|^k+a_{10}x+\dots+a_{r0}x^r+o_2(x^r).$$

Take $h(y)=a_{01}y$ and $g(x)=f(x,|x|^k)-h(|x|^k)$. Clearly, h is r times differentiable and g is r times differentiable outside 0. We also have $g(x)=a_{00}+a_{10}x+\dots+a_{r0}x^r+o_2(x^r)$ when $x\to 0$. So g is r times differentiable also at 0.

Next we prove that $V=V_1$ is not r times differentiably basic for each $1<r$. Define a differentiable function $f:V\to\mathbf{R}$ by $f(x,y)=xy$, where $y=|x|$. If V is r times differentiably basic for some $r\ge 2$, then there are r times differentiable functions

$$g,h:\mathbf{R}\to\mathbf{R}\quad\text{such that}\quad f(x,|x|)=x|x|=g(x)+h(|x|).$$

Hence $g(x)-g(-x)=2x^2$ for $x\in[0;1]$. But this is impossible because g is 2 times differentiable, hence for $x\to+0$

$$g(x)=g(0)+ax+bx^2+o(x^2)\quad\text{and}\quad g(-x)=g(0)-ax+bx^2+o(x^2).$$

At last we prove that V_k is not r times differentiably basic for k odd and each $k<r$. Define a differentiable function $f:V_k\to\mathbf{R}$ by $f(x,y)=xy$, where $y=|x|^k$. If V is r times differentiably basic for some $r>k$, then there are r times differentiable functions

$$g,h:\mathbf{R}\to\mathbf{R}\quad\text{such that}\quad f(x,|x|^k)=x|x|^k=g(x)+h(|x|^k).$$

Hence $g(x)-g(-x)=2x^{k+1}$ for each $x\in[0;1]$. But this is impossible for k odd because g is $(k+1)$ times differentiable, hence for $x\to+0$

$$g(x)=g_0+g_1x+\dots+g_{k+1}x^{k+1}+o(x^{k+1})\quad\text{and}\quad g(-x)=g_0-g_1x+\dots+g_{k+1}x^{k+1}+o(x^{k+1}).\qquad\square$$

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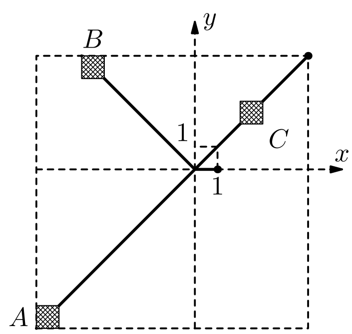

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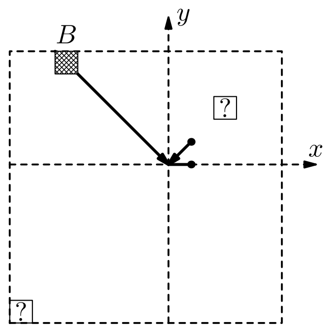

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