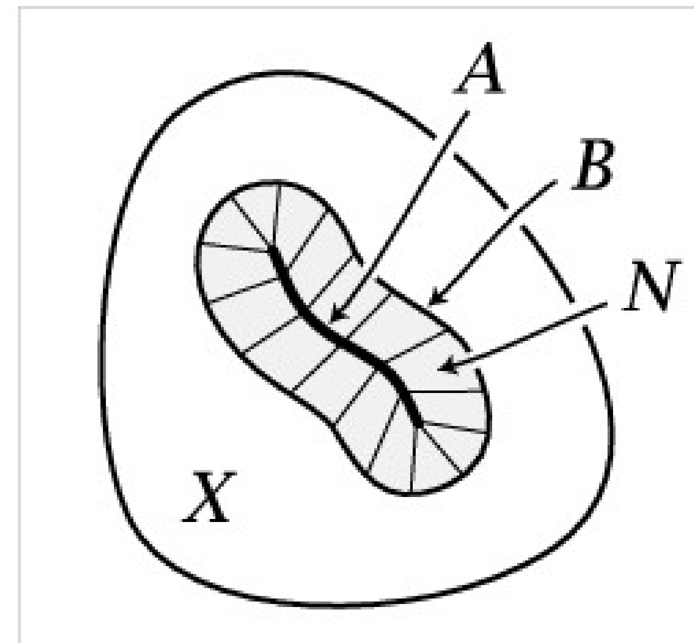
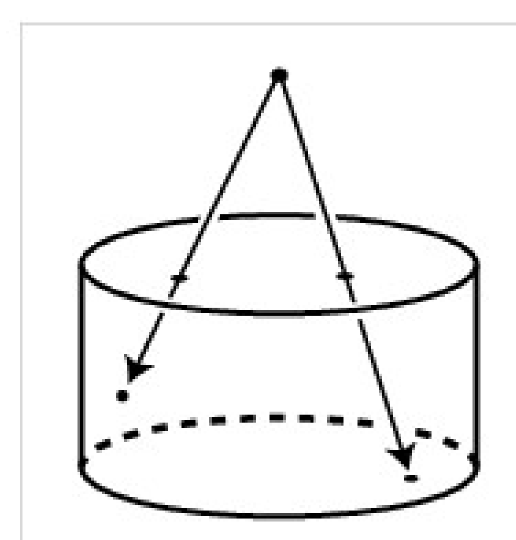
homotopy extension here can be attributed to the bad structure of (X, A) near 0. With nicer local structure the homotopy extension property does hold, as the next example shows.

Example 0.15. A pair (X, A) has the homotopy extension property if A has a mapping cylinder neighborhood in X, by which we mean a closed neighborhood N containing a subspace B, thought of as the boundary of N, with $N - B$ an open neighborhood of A, such that there exists a map $f : B \to A$ and a homeomorphism $h : M_f \to N$ with $h | A \cup B = 1\!\!1$. Mapping cylinder neighborhoods like this occur fairly often. For example, the thick letters discussed at the beginning of the chapter provide such neighborhoods of the thin letters, regarded as subspaces of the plane. To verify the homotopy extension property, notice first that $I \times I$ retracts onto $I \times \{0\} \cup \partial I \times I$, hence $B \times I \times I$ retracts onto $B \times I \times \{0\} \cup B \times \partial I \times I$, and this retraction induces a retraction of $M_f \times I$ onto $M_f \times \{0\} \cup (A \cup B) \times I$. Thus $(M_f, A \cup B)$ has the homotopy extension property. Hence so does the homeomorphic pair $(N, A \cup B)$. Now given a map $X \to Y$ and a homotopy of its restriction to A, we can take the constant homotopy on $X - (N - B)$ and then extend over N by applying the homotopy extension property for $(N, A \cup B)$ to the given homotopy on A and the constant homotopy on B.



Proposition 0.16. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.*

Proof: There is a retraction $r : D^n \times I \to D^n \times \{0\} \cup \partial D^n \times I$, for example the radial projection from the point $(0, 2) \in D^n \times \mathbb{R}$. Then setting $r_t = tr + (1 - t)1\!\!1$ gives a deformation retraction of $D^n \times I$ onto $D^n \times \{0\} \cup \partial D^n \times I$. This deformation retraction gives rise to a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ since $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$. If we perform the deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ during the t-interval $[1/2^{n+1}, 1/2^n]$, this infinite concatenation of homotopies is a deformation retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$. There is no problem with continuity of this deformation retraction at $t = 0$ since it is continuous on $X^n \times I$, being stationary there during the t-interval $[0, 1/2^{n+1}]$, and CW complexes have the weak topology with respect to their skeleta so a map is continuous iff its restriction to each skeleton is continuous. ☐



Now we can prove a generalization of the earlier assertion that collapsing a contractible subcomplex is a homotopy equivalence.

Proposition 0.17. *If the pair (X, A) satisfies the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*