Probability Theory – 2015

Class 5: Pairs of random variables

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Outline

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Joint cumulative distribution function

The joint cumulative distribution function (joint CDF) of a pair of random variables X and Y is

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

Properties of joint CDF

For a pair of random variables X and Y:

- $0 \leq F_{X,Y}(x,y) \leq 1$
- $F_X(x) = F_{X,Y}(x,\infty)$
- $F_Y(y) = F_{X,Y}(\infty,y)$
- $F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = 0$
- If $x \leq x_1$ and $y \leq y_1$, then

$$F_{X,Y}(x,y) \leq F_{X,Y}(x_1,y_1)$$

- $F_{X,Y}(\infty,\infty) = 1$

Furthermore, we have

$$P(x_1 < X \leq x_2,\ y_1 < Y \leq y_2) = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$$

Joint probability mass function

The joint probability mass function (joint PMF) of the discrete random variables X and Y is:

$$P_{X,Y}(x, y) = P(X = x, Y = y).$$

The range of the pair of random variables X and Y is defined as

$$S_{X,Y} = \{(x, y) \mid P(X = x, Y = y) > 0\}.$$

For a set B in the (X, Y)-plane we have

$$P(B) = \sum_{(x,y)\in B} P_{X,Y}(x, y).$$

Joint probability density function

The joint probability density function (joint PDF) $f_{X,Y}$ of the random variables X and Y is a function such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v) \, \mathrm{d}v \mathrm{d}u$$

Consequence:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Interpretation:

$$f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y = P(x < X \le x + \mathrm{d}x, y < Y \le y + \mathrm{d}y)$$

Properties of joint PDF

A joint probability density function $f_{X,Y}(x,y)$ has the following properties:

- $f_{X,Y}(x,y) \geq 0$ for all (x,y),
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x = 1.$

Furthermore, we have

$$P(A) = \iint_{(x,y)\in A} f_{X,Y}(x,y)\,\mathrm{d}y\mathrm{d}x$$

Example

Given are two random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the joint cumulative distribution function.

Example (continued)

Given are two random variables X and Y with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the probability $P(A) = P(X + Y \le 1)$.

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Marginal probability mass function

Let X and Y be discrete random variables with joint probability mass function $P_{X,Y}(x,y)$. Then we have

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y), \quad P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x,y).$$

These functions are called the marginal probability mass functions (marginal PMF's) of the random variables X and Y.

Marginal probability density function

Let X and Y be continuous random variables with joint probability density function $f_{X,Y}(x,y)$. Then we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x.$$

These functions are called the marginal probability density functions (marginal PDF's) of the random variables X and Y.

Example

Given two random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{5}{4}y, & -1 \le x \le 1, \;\; x^2 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

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Functions of two random variables

For discrete random variables X and Y, the function $W = g(X, Y)$ has probability mass function

$$P_W(w) = \sum_{\substack{(x,y) \\ g(x,y)=w}} P_{X,Y}(x, y).$$

Functions of two random variables

For continuous random variables X and Y, the function $W = g(X, Y)$ has cumulative distribution function

$$F_W(w) = P(W \leq w) = \iint_{g(x,y) \leq w} f_{X,Y}(x, y)\,\mathrm{d}y\mathrm{d}x.$$

For $W = \max(X, Y)$ we have

$$F_W(w) = \int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x, y)\,\mathrm{d}y\mathrm{d}x.$$

Example

Given are two random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)}, & x \geq 0, \;\; y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the probability density function of $W = Y/X$.

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Expectation

For random variables X and Y, the expectation of $W = g(X, Y)$ is

Discrete: $$E(W) = \sum_{x \in S_x} \sum_{y \in S_y} g(x, y) P_{X,Y}(x, y),$$

Continuous: $$E(W) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, \mathrm{d}y \mathrm{d}x.$$

Expectation

We have

$$E(g_1(X,Y)+\cdots+g_n(X,Y)) = E(g_1(X,Y))+\cdots+E(g_n(X,Y)).$$

In particular:

$$E(X+Y) = E(X)+E(Y).$$

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Covariance

The variance of the sum of two random variables is:

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2E[(X-\mu_X)(Y-\mu_Y)].$$

The covariance of the random variables X and Y is:

$$\text{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)].$$

We call the random variables X and Y uncorrelated if $\text{Cov}[X,Y]=0$.

We have

- $\text{Cov}(X, Y) = E(XY) - \mu_X \mu_Y$,
- $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X, Y]$,
- If $X = Y$, then$\text{Cov}(X, Y) = \text{Var}[X] = \text{Var}[Y]$.

Correlation coefficient

The correlation coefficient of two random variables X and Y is:

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y}.$$

We always have $-1 \leq \rho_{X,Y} \leq 1$.

If $Y = aX + b$, then

$$\rho_{X,Y} = \begin{cases} -1, & \text{if } a < 0, \\ 0, & \text{if } a = 0, \\ 1, & \text{if } a > 0. \end{cases}$$

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Conditional joint PMF

For discrete random variables X and Y and an event B with $P[B] > 0$, the conditional joint PMF of X and Y given B is

$$P_{X,Y|B}(x,y) = P(X = x, Y = y|B).$$

We have

$$P_{X,Y|B}(x,y) = \begin{cases} \frac{P_{X,Y}(x,y)}{P(B)}, & \text{if } (x,y) \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Conditional joint PDF

For contintuous random variables X and Y and an event B with $P[B] > 0$, the conditional joint PDF of X and Y given B is

$$f_{X,Y|B}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P(B)}, & \text{if } (x,y) \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Example

Given random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15}, & 0 \leq x \leq 5, \ \ 0 \leq y \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the conditional joint PDF of X and Y given

$$B = \{X + Y \geq 4\}.$$

Conditional expectation

For random variables X and Y and an event B with $P(B) > 0$, the conditional expectation of $W = g(X, Y)$ given B is:

Discrete:
$$E[W|B] = \sum_{x \in S_x} \sum_{y \in S_y} g(x, y) P_{X,Y|B}(x, y)$$

Continuous:
$$E[W|B] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y|B}(x, y) \, \mathrm{d}y \mathrm{d}x$$

Example (continued)

Given random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15}, & 0 \leq x \leq 5, \ \ 0 \leq y \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the conditional expectation of $W = X + Y$ given

$$B = \{X + Y \geq 4\}.$$

Outline

Conditional PMF

For discrete random variables X and Y and an event $Y = y$ with $P_Y(y) > 0$, the conditional PMF of X given $Y = y$ is

$$P_{X|Y}(x|y) = P(X = x|Y = y).$$

We have

$$P_{X,Y}(x,y) = P_{X|Y}(x|y)P_Y(y) = P_{Y|X}(y|x)P_X(x).$$

Conditional expectation

Let X and Y be discrete random variables. For any $y \in S_y$, the conditional expectation of $g(X, Y)$ given $Y = y$ is

$$E[g(X, Y)|Y = y] = \sum_{x \in S_X} g(x, y) P_{X|Y}(x|y).$$

In particular:

$$E[X|Y = y] = \sum_{x \in S_X} x P_{X|Y}(x|y).$$

Conditional PDF

Let X and Y be continuous random variables. For y such that $f_Y(y) > 0$, the conditional PDF of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

We have

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x).$$

Example

Given are random variables X and Y with joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the conditional probability density function of X given Y.

Conditional expectation

For continuous random variables X and Y and an event $Y = y$ with $f_Y(y) > 0$, the conditional expectation of $g(X, Y)$ given $Y = y$ is

$$E[g(X, Y)|Y = y] = \int_{-\infty}^{\infty} g(x, y)\, f_{X|Y}(x|y)\, \mathrm{d}x.$$

In particular:

$$E[X|Y = y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, \mathrm{d}x.$$

Conditional expectation

The conditional expectation $E[X|Y]$ is a function of random variabele Y such that if the realization of the random variable Y is equal to y, then the realization of the random variable $E[X|Y]$ is equal to $E[X|Y=y]$.

We have

$$E[E[X|Y]] = E[X],$$

$$E[E[g(X)|Y]] = E[g(X)].$$

Outline

Independent random variables

Random variables X and Y are independent if and only if:

Discrete: $P_{X,Y}(x,y) = P_X(x)P_Y(y)$

Continuous: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

Example

Given are random variables U and V with joint PDF

$$f_{U,V}(u,v) = \begin{cases} 24uv, & u \geq 0,\ v \geq 0,\ u+v \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Are the random variables U and V independent?

Properties of independent random variables

For independent random variables X and Y:

- $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$,
- $E(XY) = E(X)E(Y)$,
- $\text{Cov}(X, Y) = \rho_{X,Y} = 0$,
- $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$,
- $E(X|Y = y) = E(X)$ for all $y \in S_Y$,
- $E(Y|X = x) = E(Y)$ for all $x \in S_X$.

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Bivariate Gaussian PDF

Random variables X and Y have a bivariate Gaussian PDF with parameters μ_1, σ_1, μ_2, σ_2, and ρ if

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]$$

with μ_1 and μ_2 arbitrary real numbers, $\sigma_1 > 0$ and $\sigma_2 > 0$ and $-1 < \rho < 1$.

Interpreation of the parameters

We have

- $E(X) = \mu_1$,
- $\text{Var}[X] = \sigma_1^2$,
- $E(Y) = \mu_2$,
- $\text{Var}[Y] = \sigma_2^2$,
- $\rho_{X,Y} = \rho$,
- $\text{Cov}(X, Y) = \rho\sigma_1\sigma_2$.

Marginals of bivariate Gaussian random varaibles

If X and Y are bivariate Gaussian random variables, then X is a Gaussian (μ_1, σ_1) random variable and Y is a Gaussian (μ_2, σ_2) random variable:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-(x-\mu_1)^2/(2\sigma_1^2)},$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-(y-\mu_2)^2/(2\sigma_2^2)}.$$

Conditional PDF's of bivariate Gaussian random varaibles

If X and Y are bivariate Gaussian random variables, then the conditional PDF of X given $Y = y$ is equal to

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\tilde{\sigma}_1^2}} e^{-(x-\tilde{\mu}_1(y))^2/(2\tilde{\sigma}_1^2)}$$

with

$$\tilde{\mu}_1(y) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y - \mu_2), \quad \tilde{\sigma}_1^2 = \sigma_1^2(1 - \rho^2).$$

Remark: Bivariate Gaussian random variables X and Y are uncorrelated if and only if X and Y are independent!

Outline

Summary

- We discussed the joint CDF, joint PMF and joint PDF of a pair of random variables.
- We showed how to obtain the marginal PMF's (or marginal PDF's) of the two random variables from their joint PMF (or joint PDF).
- We studied functions of a pair of random variables.
- We defined covariance and correlation coefficient.
- We looked at conditioning by an event and by a random variable.
- We defined independence between two random variables.
- We studied bivariate Gaussian random variables.