The University of Melbourne

Department of Mathematics and Statistics

Semester One Assessment 2001

620–141 Mathematics A

Examination duration: Three hours
Reading time allowed: Fifteen minutes

This paper has five pages including this page.

Authorized Materials: No materials are authorized. Students entering the examination venue with notes or printed material related to the subject, calculators or computers, or mobile phones, should stand in their place immediately and surrender these to an invigilator before the instruction to commence writing is given.

Instructions to Invigilators: No special materials are to be supplied. This examination paper may be removed at the conclusion of the examination.

Instructions to Students:

- Write all your solutions in the booklets provided.
- Number the questions clearly, and start each question on a new page.
- Use the *left* pages for rough working. Write material you wish to be marked on *right* pages only.
- Read each question completely before starting to answer it.
- You should attempt as many questions as time permits.
- Each question carries the same number of marks.

Question 1. Solve the following system by using row operations on an appropriate augmented matrix:

$$\begin{aligned} 2x + 4y + 3z &= -1 \\ -x - 3y - 2z &= 1 \\ 3x + y + z &= 2 \end{aligned}$$

Question 2. The augmented matrix for a system of linear equations in the variables x, y and z has been reduced by row operations to the following row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & a-2 & 2 \\ 0 & 0 & 2-2a & b-1 \end{array}\right]$$

For which values of a and b does the system have

(i) no solutions;

(ii) exactly one solution;

(iii) infinitely many solutions. For these values of a and b write down all solutions of the system.

Question 3. (i) Given that

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 8 & -2 \\ 0 & 0 & 6 \\ 12 & -12 & 3 \end{bmatrix},$$

evaluate AB.

(ii) Use your answer in (i) to find the 3×3 matrix C such that

$$CA = \begin{bmatrix} 3 & 4 & 2 \\ 3 & -1 & 4 \\ 6 & 2 & 0 \end{bmatrix}.$$

Question 4. (i) Compute the determinant of the following matrix:

$$M = \begin{bmatrix} 3 & 3 & 6 & 6 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

(ii) Use your answer in (i) to compute $\det(2M)$.

Question 5. Let $\mathbf{a} = (1, 3, 0)$, $\mathbf{b} = (2, -1, 3)$ and $\mathbf{c} = (1, 2, -1)$.

(i) Find the cosine of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

(ii) Find $\mathbf{a} \times \mathbf{b}$ and $(\mathbf{a} \times \mathbf{c}).\mathbf{b}$.

(iii) Find the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$ and the volume of the parallelepiped (parallel sided box) defined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Question 6. (i) Write down the cartesian equation(s) of the line L through the point P with co-ordinates (2,1,0) which is perpendicular (normal) to the plane Π with equation $2x + y - z = -1$.

(ii) Find where the line found in (i) meets the plane Π and hence find the distance between the point P and Π.

Question 7. Find the cartesian equation for the plane passing through the points with co-ordinates $(2, -1, 1)$, $(1, 0, 0)$ and $(1, 2, -1)$. Find, in parametric form, the equation of the line of intersection of this plane with the plane perpendicular to the vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and passing through the point with co-ordinates $(1, 2, 3)$.

Question 8. (i) Define $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

(ii) Solve for x the equation

$$3\sinh(x) - 2\cosh(x) = 2 \ .$$

Question 9. Find the equation of the tangent to the curve

$$x^2 + x\sinh(y) + y^2 = 9$$

at the point $(3, 0)$.

Question 10. Express

$$f(x) = \cos(\arctan(x)) \ , \quad -\infty < x < \infty$$

as an algebraic function of x.

Question 11. Find the global maximum value and the global minimum value of g, where

$$g(x) = 2x^5 - 5x^4 + 1 \ ,$$

on the interval $1 \leq x \leq 3$.

Question 12. Let $f(x) = \arctan(x)$.

Find the first and second derivatives of f, and state the intervals on which

(i) f is increasing;

(ii) f is concave up.

Question 13. In this question you may assume that

$$1 - \tanh^2(x) = \text{sech}^2(x) \ .$$

(i) Find the derivative of $\dfrac{\sinh(x)}{\cosh(x)} = \tanh(x)$.

(ii) Find the derivative of $\text{arctanh}(x)$.

(iii) Find, for $0 < x < 1$, the derivative of $\text{arctanh}(\sqrt{1-x^2})$.

Question 14. Suppose

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 - 2y + 4 \\ \text{and} \quad v(x,y) &= 3x^2y - y^3 + 2x \end{aligned} \ .$$

(i) Show that $\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0$.

(ii) Evaluate $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$.

(iii) Evaluate $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

(iv) Evaluate $\dfrac{\partial^2 u}{\partial x \partial y}$.

Question 15. Find the equation of the tangent plane to the surface

$$z = x^2 + 3xy - y^2 + x + 5$$

at the point $(1, 0, 7)$.

Question 16. Find the direction in which the function

$$f(x, y) = x \sin(y)$$

decreases fastest, at the point $(x, y) = (3, \frac{\pi}{6})$.

Question 17. Find all stationary points of the function

$$f(x, y) = 2x^4 + y^2 - 12xy + 3 \,.$$

For each stationary point, state whether it is a maximum, minimum or saddle point.

Question 18. Simplify the following expressions involving complex numbers:

(i) $\operatorname{Im}\left(\dfrac{1}{3 - 2i\sqrt{5}}\right)$; (ii) $\left|\dfrac{(2 + i\sqrt{3})\overline{(1 - i\sqrt{2})}}{3 - i\sqrt{2}}\right|$;

(iii) $\left(\dfrac{1 - i}{\sqrt{2}}\right)^{79}$; (iv) $\sinh\left(\dfrac{\pi i}{6}\right)$.

Question 19. Use the complex exponential to evaluate the following:

(i) $\dfrac{d^6}{dt^6}\left(e^{2t} \cos 2t\right)$; (ii) $\displaystyle\int e^{2t} \sin 3t \, dt$.

Question 20. Expressing your answers in cartesian form, find the five roots of the polynomial equation

$$z^5 - z^4 + z^3 - z^2 + z - 1 = 0 \,,$$

and plot them as points in the complex plane.